L&PS – Logic and Philosophy of Science
Vol. IX, No. 1, 2011, pp. 103-110

The birth of the non-Euclidean geometries As the more significant crisis in the foundations of modern Mathematics

Antonino Drago
University of Pisa
drago@unina.it

1. The two foundational options in Lobachevsky's non-Euclidean geometry.
2. The several programs on the foundations of Mathematics
3. The main conflicts generated by incommensurability phenomena
4. Searching FoM in spite of incommensurabilities
5. Conclusion: Back to Lobachevsky!

ABSTRACT. The unsuccessful effort for circumscribing the foundations of Mathematics by the programs launched around the year 1900 is rationally re-constructed by means of the two basic options which emerged from Lobachevsky's work on non-Euclidean geometry. It is proved that the two main programs, Hilbert's and Brouwer's, implicitly wanted to achieve just these options; which however imply unavoidable incommensurability phenomena, which barred their programs. As a conclusion, one century and half ago the birth of non-Euclidean geometries suggested much more than new axioms on parallel lines, rather the very foundations of Mathematics.

Published by L&PS – Logic and Philosophy of Science
http://www2.units.it/ ~episteme — ISSN: 1826-1043

1. The two foundational options in Lobachevsky's non-Euclidean geometry

Surely, the birth of the non-Euclidean geometry (NEG) caused a great crisis in the Foundations of Mathematics (FoM) and implied great changes also in the relationship of Mathematics with the physical sciences(Kline 1979, 861, 879)[1].

An historical analysis on the works by the main founder of NEG, Lobachevsky, shows that never he started a geometrical theory in a deductive way from some axioms. His main work, "written by a master hand"(Gauss 1846), manifests an alternative organisation of a theory, that is aimed to formally solve two problems - i.e. a new definition of parallelism and the value of the parallelism angle -, through two *ad absurdum* proofs. I call this kind of organisation a problem-based organization (PO) in alternative to the axiomatic one (AO)[2].

Moreover, Lobachevsky (1845) stated that he chose the potential infinity (PI), instead of the actual infinity (AI), which in his time dominated the development of Mathematics through the infinitesimal analysis. In fact, he re-founded the geometrical theory in an operative way, through (no more ruler and compass but) the cut operation[3]. Moreover, he wrote a book reiterating all basic notions of calculus by means of finite algorithms only (Lobachevsky 1834).

---

[1] Actually, around the time of French revolution, hence before the birth of NEGs, some new theories born: both L. Carnot (geometry and calculus) and Lagrange (calculus) suggested the same alternative choices. But these theories either failed (the last one) or have been ignored as an incomplete theory (the first one) or as a too naïve theory (the second one). The novelty of the complete theories of NEG, radically changing the basic notion of space made the crisis unavoidable.

[2] Some more scientific theories exhibit this kind organisation, although in a less accurate way: L. Carnot's calculus and geometry, L. Carnot's mechanics (in each edition of his book two pages illustrate both kinds of organisation), Lagrange's mechanics, Avogadro's atomic theory, S. Carnot's thermodynamics, Galois' group theory, Klein's Erlanger program, Planck's quantum theory, Einstein's special relativity, Kolmogorov's foundations of minimal logic, Markov's theory of computable functions. Moreover, both Poincaré and Einstein stressed that previous physical theories exhibit two kinds of organisation which correspond to the above ones (Drago 2005).

[3] Actually, he was inconsistent in two occasions; when defining anew the parallel line, he implicitly referred to an infinitesimal; moreover, he considered the figure cut at the infinity by an asymptotic orthoscheme (Bazhanov and Drago).

2. The several programs on the foundations of Mathematics

Before the NEG mathematicians conceived the FoM through some basic notions (ruler and compass, infinitesimals, etc.) or a basic theory (geometry, calculus, etc.); at most, they tried to qualify the mathematical method by debating about analysis and synthesis – but unsuccessfully. The birth of the NEGs dramatically made uncertain the mathematical bases, in particular, the two basic notions of both space and axiom.

In order to answer to this crisis, some mathematicians, by ignoring the above two options, suggested programs of research claiming that some specific tools will be capable to circumscribe FoM. All together these programs constituted a great effort of mathematicians' community also because they born independently from all past philosophical systems, and without any help from contemporary philosophers.

Actually, Cantor founded no more than one theory; but he planned to found anew, directly from his theory, all other mathematical theories. Similar considerations can be reiterated for Frege's logicist program, which wanted to draw the whole body of mathematical theories from one theory, mathematical Logic.

In a first time of his search for FoM, also Hilbert focused his attention upon one theory, i.e. Euclidean geometry. However, he re-formulated it as a formally axiomatic theory; then he abstracted from this experience a general program, i.e. to reiterate, in every scientific theory, the same work of axiomatization he had performed in geometry. Brouwer too started from one theory, arithmetic; yet he founded it according to a new method promising to found anew a great part of the body of mathematical theories.

Remarkably, all these programs chose AO – Cantor and Frege wanted to draw all theories from respectively the basic notion of a set, and the basic laws of Logic; mainly Hilbert chose (and improved) AO -, except for Brouwer who suggested both a mistrust in the logical axioms of an OA (Brouwer: 1908). and some hints for founding a non-classical logic, which surely does not manage the deductions of an AO. Moreover, all programs chose AI –Frege disregarded PI, Cantor wanted to capture it through the notion of a set, and Hilbert claimed as unavoidable the "ideal elements" (we read: AI) – except for again Brouwer, who just opposed to AI for choosing PI, which he characterised almost exactly.

Hence, these programs, all together, represented an implicit tension to recognise the complex of the four choices already manifested by Lobachevsky's work.

3. The main conflicts generated by incommensurability phenomena

It is not a case that, the two programs formally developing the old idea of considering one basic theory as constituting the FoM, i.e. both Cantor's and Frege's programs, failed few years after their births. A single theory was no more enough to embrace all theories; it unavoidably meets contradictions (say, Russell's antinomies) because for this so wide scope either its main notion has to be a semi-philosophical one, as a set is[4], or the laws of the thought cannot directly refer to mathematical beings.

Remarkably, both the remaining programs, i.e. Hilbert's program and Brouwer's program, *declared their basic choices, although without qualifying them as choices*. Hilbert's program declared both the "axiomatics" and the "ideal elements" as characterising the FoM. On the other hand, Brouwer declared the basic choice PI and, as above mentioned, explicitly suggested some elements of PO.

Thus, each founder of the two remained programs separately recognized a different pair of opposite choices. But, by ignoring the totality of the four choices, he considered the development of his program as an exclusive Truth, i.e. as excluding the validity of a program based on the opposite pair of choices.

Indeed, the mathematicians did not apperceived a new phenomenon generated by the mutual comparison of two programs differing in their basic choices. The two dichotomic choices about each option are mutually incompatible; e.g. the mathematical objects including AI (e.g., Zermelo's axiom) are unthinkable by the mathematics bounded to use IP only; on the other hand, the undecidable phenomena generated by PI are invisible by a mathematics making use of AI. Evenly, an AO theory solves whatsoever problem by means of an *a priori* fixed method; i.e. it deductively draws a long list of theorems, including the decisive one. On the other hand, a PO theory is based upon an apparently unsolvable problem; by appealing to common knowledge only, it looks for a new scientific method, capable to solve this basic problem. In addition, a PO theory, being of an inductive kind, cannot share classical logic. Hence, the respective kinds of logic of AO and PO in no way are implied one by another.[5]

Owing to the radical divergences implied by the different choices, two theories or two programs differing in at least a basic choice are called mutually in-

---

4 However, Cantor's theory was preserved through an ambitious operation which changed it in ZFC set theory, whose choices are the same of previous theory.

5 Let us remember the studies by Kolmogoroff, Glyvenko and Goedel, on the no more than partial translations between classical logic and the best instance of non-classical logic, the intuitionist one.

commensurable (Drago 1987)[6]. In this case an easy translation from one theory to another one does not exist, since the meanings of their basic notions suffer radical variations. E.g. the notion of infinity presents two meanings, either AI or PI; in fact, they generated a great debate, started by Kronecker and Poincaré, then drastically dramatised by both Brouwer and Hilbert. Eventually, this debate resulted to be inconclusive, being mathematicians unable to recognize in this notion an option. One more case is the notion of space, which too originated a great debate after the birth of NEGs.

We see that ironically the progress achieved by the two main programs, Hilbert's (AI&AO) and Brouwer's (PI&PO), i.e. to have together implicitly recognised the FoM by directly taking in account each a different pair of basic choices, resulted in a mutual incommensurability, without a mutual translation of the meanings of their basic notions.

As an historical fact, Hilbert and Brouwer entered in a harsh struggle. Brouwer started a sharp contrast with mathematicians' community by rejecting a great part of the basic mathematical notions and Hilbert's program too. In retort, Hilbert, being supported by his exceptional command on the whole body of Mathematics, opposed an exclusive attitude about FoM; not only he rejected the intuitive part of Mathematics – just what Brouwer wanted to develop – but, in order to both preserve and improve the past dominant choices AI&AO, eventually went to act in an hostile way against Brouwer.

Neither mathematicians nor historians have been capable to qualify the very nature of this conflict generated by the comparison of two incommensurable programs.

4. Searching FoM in spite of incommensurabilities

However, each of these programs of research went, although through a non-planned, slow and obscure work, to include the pair of the opposite choices, so that they approached near to the entire knowledge of the FoM.

In order to answer to some criticisms, Hilbert added to his program a metamathematics; whose arithmetic theory Hilbert declared to be "finitist"; hence, *grosso modo* PI. Moreover, he admitted that the organisation of metamathematics could not be an AO (otherwise, a *regressus ad infinitum* occurs on the consistency of its axioms); rather, it was aimed to solve a basic problem, i.e. to discover a new

---

[6] This definition formalises the intuitive notion suggested by both Kuhn's and Feyerabend; see for ex. (Kuhn1969).

method for proving arithmetic's consistency; hence, the metamatemathics is nothing else than PO[7]. In sum, Hilbert's program involved even its alternative choices, i.e. PI and PO, though in a covert way and by putting them in a subordinate role to the previous ones.

As a consequence, Hilbert's program had to compare metamathematics PI&PO with the mathematical theory under scrutiny (arithmetic) to which he attributed the choices AI&AO. Since these pairs of choices are different, the wanted comparison implied an incommensurability phenomenon. In fact, in advancing his program, a so great mathematician as Hilbert worked in an inappropriate way[8] and eventually in an unsuccessful way.

Goedel's genius was to conceive a theorem formalising this comparison, pertaining to a more higher level of analysis than a single mathematical theory. It proved that a full comparison of arithmetic with its metamathematics is impossible, owing to essential undecidabilities[9]. This result barred Hilbert's program.

Remarkably, although this result constituted the most advanced result of Hilbert's program, it added a merely negative knowledge on the FoM as supposed by Hilbert. Rather, it added as a subordinate result the first formal definition of recursive functions, i.e. ironically the mathematics of basic choice of the alternative program, PI. Given this dual role played by Goedel's result, no surprise if subsequently it constituted a stumbling block to all efforts for improving the original Hilbert's program.

On the other hand, Brouwer's program was unable to re-construct a relevant part of the body of Mathematical theories, if not by including some notions surpassing PI, i.e. choice sequences and fixed-point theorem. After him, Heyting added one more compromise; he axiomatized intuitionist logic in AO, by merely adding a verbal *proviso* about the insufficiency of any axiomatic to trustfully grasp the intuitive theory (Heyting 1960). Hence, this program too enlarged its

---

[7] Moreover, after a Brouwer's criticism, Hilbert admitted that in metamathematics the logical law of the excluded middle fails; being weaker than classical one, its logic is thus nothing else than the intuitionist one; which may be the logic of a PO.

[8] See the harsh appraisal by H. Freudenthal (1972) on Hilbert as a philosopher of science.

[9] I attempted a new interpretation of both the historical meaning and the thesis of the first Goedel's theorem (Drago 1993). The comparison is possible when one changes the choices of metamathematics, as Gentzen did by introducing in it the transfinite induction (AI).

scope by sharing the opposite choices AI and AO case by case; so that to produce independent theories.

Among the programs launched around the year 1900, no one was successful with respect to its original aims. But this historical experience was not a mere failure, because two out of the four starting programs - Hilbert's and Brouwer's -, approached near to the recognition of the FoM,, although ironically no one was capable to explicitly recognise the opposite foundations of the other program, and hence all the four choices in an objective way. Given this unattained goal, no surprise if the final result of the fight between these two programs was a no-contest situation.

After Goedel's theorem, the mathematicians, by apperceiving the FoM in an incomplete and obscure way, eventually reached an odd agreement. They recognized (all the choices claimed by) both programs as representing a more comprehensive viewpoint than previous Hilbert's program; i.e. an abundance of techniques for independently developing Mathematics. But, since all these techniques generated through their different basic choices even more incommensurabilities; which barred any attempt for advancing both the knowledge of FoM and the awareness about the past history. It is not surprising that this agreement not only frustrated a further search for knowing FoM, rather supported Bourbaki's suggestion to renounce to this research.

5. Conclusion: Back to Lobachevsky!

In retrospect, this history was caused by a refractoriness to recognise the essential pluralism born through Lobachevsky's NEG, which manifested the alternative choices PI and PO to the well known AI and AO. By ignoring this pluralism, mathematicians insisted to see the FoM as constituted by one theory (Kronecker, Frege, Cantor), or one choice (Hilbert's program in a first time), or two choices (Brouwer's program, Hilbert's second program) or in an ill-defined set of all choices (both the two advanced programs and moreover the present time appraisals on FoM).

The above successful interpretation of the cumbersome and deceiving research on FoM along the past century proves that the above four of basic choices constitute the very FoM. Hence, the time is came to recognize the birth of Lobachevsky's theory as the decisive event for both the manifestation of the FoM and the beginnings of a pluralist history of Mathematics, previously misinterpreted by both Hilbert and Brouwer as a win-lose competition and by post-Goedel mathematicians as an indistinct ecumenism.

BIBLIOGRAPHY

BAZHANOV V. and DRAGO A. (2010): ”A logical analysis of Lobachevsky’s geometrical theory”, *Atti Fond. Ronchi*, 64, p. 453-481.

BROUWER L.E.W. (1908): 1908C in *Collected Works*, vol. I, Amsterdam: North-Holland 1985, p. 107-111.

DRAGO A.: "An effective definition of incommensurability", comm. to *VIII Congress on Logic, Methodology and Phil. Sci.*, Moscow, 1987, 4, pt.1, 159-162 and in CELLUCCI C. et al. (eds.) 1988: *Temi e prospettive della logica e della filosofia della scienza contemporanea*, Bologna: CLUEB vol. II, 117-120.

DRAGO A. (1993): “Is Goedel's incompleteness theorem a consequence of the two kinds of organization of a scientific theory?”, in WOLKOWSKY Z.W.K. (ed.): *First International Symposium on Goedel's Theorems*, London: World Scientific, 107-135.

DRAGO A. (2005): “A.N. Kolmogoroff and the Relevance of the Double Negation Law in Science”, in G. SICA (ed.): *Essays on the Foundations of Mathematics and Logic*, Milano: Polimetrica 2005, p. 57-81.

FREUDENTHAL H. (1972): “Hilbert David”, in GILLISPIE C.C. (ed.): *Dictionary of Scientific Biography*, New York: Scribner’s 1972, 393 I.

GAUSS F. (1846): “Letter to Schumacher”, 28/11/1846.

HEYTING A.: *Intuitionism*, Amsterdam: North-Holland 1960, p. 106.

KLINE M. (1979): *The Mathematical Thought from the Ancient to Modern Times*, Oxford: Oxford U.P. p. 861, 879.

KUHN T.S. (1969): *The Structure of Scientific Revolutions*, Chicago: Chicago U.P., ch. 11.

LOBACHEVSKY N. I. (1834): *Algebra or Calculus of Finites* (in Russian), Kazan.

LOBACHEVSKY N. I. (1845): “Introduction”, *New Principles of Geometry* (in Russian; Engl. tr.: Austin, 1897).