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# Sheep without SOL
# The Case of Second-Order Logic

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ABSTRACT. According to Quine, second-order logic (SOL) is set-theory in disguise. This claim has been disputed on solid grounds, in particular in the work by George Boolos on plural quantification. Nevertheless, since plural logic (PL) and SOL are equi-interpretable, they seem to provide equal alternatives. The picking of one over the other seems to rely merely on ontological (or at least broadly philosophical) preferences. In the present article, I am going to address a non-ontological argument for a distinction between PL and SOL. This argument will be grounded on the different mathematical applicability to set-theory that PL and SOL respectively show to have.

## 1. Overview

According to Quine, second-order logic (SOL) is set-theory in disguise.[1] Nevertheless, this claim has been disputed by Boolos' plural interpretation of se-

[1] The correct quotation is "set-theory in sheep's clothing": see Quine (1986).


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cond-order quantification.[2] Boolos' semantics is not committed to the existence classes or properties: second-order variables are interpreted as varying *plurally* over the first-order individuals. Revising Tarski's semantics, Boolos provides the semantic clauses for second-order logic, where the usual function of assignment is substituted by a *one-many relation* of assignment *R*. Boolos' semantics has raised several criticisms over time,[3] but in spite of them, plural logic (PL) is nowadays accepted as a legitimate branch of logic, very useful under several respects, for instance in the philosophical analysis of the foundations of mathematics.[4]

Boolos (1984) and (1985) show that PL and SOL are equi-interpretable. So, they seem to provide equal logical alternatives, with the same consistency strength. The fact that one may be preferred over the other seems to rely merely on ontological, or at least broadly philosophical, motivations.[5]

In the present article, I will investigate this issue by applying PL and SOL to a consistent predicative Fregean set-theory and I will show that these augmentations lead to theories with very different mathematical expressiveness.[6] I will then conclude that, since PL and SOL add different expressive capacity to set-theory, they also show to be different alternatives, and choosing one over the other is not just a matter of philosophical orientation.

## 2. The Theory PV

First, consider a predicative fragment of Frege's *Grundgesetze*: the theory PV, for *Predicative V*. Its language $L_{PV}$ consists of

(1.1) infinitely many first-order variables $x, y, z, \ldots$;

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2 Boolos (1984, 1985).

3 See for instance Resnik (1980), Parsons (1990), and Linnebo (2003).

4 See Burgess (2004), Uzquiano (2003).

5 The argument for the ontological innocence of PL is a standard consideration on why to prefer PL over SOL. Also, in Burgess (2004), PL is taken to provide novel philosophical motivation for limitation of size in set-theory.

6 Notice that in this article I distinguish between second-order logic (SOL) and plural logic (PL) in the following sense: by "SOL" I mean "second-order logic with a standard set-theoretic interpretation" and by "PL" I mean "second-order logic with plural interpretation". Nowadays, it is standard to have different notations for second-order logic as opposed to plural logic, but I kept almost the same notation and differed only in interpretations in order to make my main point as clear as possible.

(1.2) infinitely many second-order variables $F, G, H, \dots$ varying over a domain of classes of first-order individuals. I am going to call these classes *predicative*;[7]
(1.3) usual logical connectives and quantifiers;
(1.4) the abstraction function {: }.

The atomic formulæ of this language are the identity formulæ of the form $x=y$, and the formulæ of the form $Fx$. Existential quantification is available for both kinds of variables. Beside the first-order variables, the terms of $L_{PV}$ are the set-terms of the form $\{x: Fx\}$. The axioms of PV include a predicative comprehension axiom

(PRC) $\exists F \forall x\, (Fx \leftrightarrow \varphi)$

where φ does not contain $F$ free nor bound predicative class variables; and a formulation of Frege's Basic Law V:

(V) $(\forall F)(\forall G)(\{x: Fx\}=\{x: Gx\} \leftrightarrow \forall x\, (Fx \leftrightarrow Gx))$.[8]

## 3. The Theory P-PV

Augment PV with PL, extending it to *Plural Predicative V* (P-PV). The language $L_{P\text{-}PV}$ is a three-sorted second-order language which adds to $L_{PV}$

(2.1) a round of plural individual variables $X, Y, Z, \dots$, that vary *plurally* over the individuals of the first-order domain;
(2.2) an existential quantifier for plural variables.

The atomic formulæ of $L_{P\text{-}PV}$ are the atomic formulæ of $L_{PV}$ augmented with the formulæ $Yx$, to be read "$x$ is among the $Y$s". Formulæ of this kind express what I may call *plural reference*. Primitive existential quantification

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[7] I use "predicative classes" since the axiom governing them (PRC below) is predicative. I will use the same label also in the two augmentations with PL and SOL, for reasons that I hope will be clear to the reader.

[8] PV is a sub-system of the consistent Heck (1996), since Heck (1996) has unrestricted schematic Basic Law V among its axioms. In Burgess (2005), Heck (1996) is shown to be equi-interpretable with Robinson arithmetic. What the first-order set-theory with primitive membership corresponding to PV is can be found in Burgess (2005, pp. 89-92).

for every kind of variables is available. Universal quantification for every kind of variables can be defined obviously. Together with the singular variables *x, y, z,…*, the *terms* of $L_{\text{P-PV}}$ are an infinite list of set-terms of the form $\{x: Fx\}$.

The semantic clauses for the formulæ of $L_{\text{P-PV}}$ will not be provided, the intuitive interpretation mentioned above being enough for my purposes. However, it is worth stressing that plural quantification is meant to be interpreted through *Boolos' plural semantics* as in Boccuni (2010), whereas predicative class variables *F, G, H, …* are meant to be interpreted as ranging over classes of first-order individuals, as in PV.

Two Comprehension Principles are available in P-PV: a plural comprehension axiom

(PLC) $\exists X \forall x\ (Xx \leftrightarrow \varphi)$

where $\varphi$ does not contain $X$ free; and a predicative comprehension axiom

(PRC*) $\exists F \forall x\ (Fx \leftrightarrow \psi)$

where $\psi$ contains neither $F$ free, nor free plural variables, nor bound predicative class variables. It may contain free predicative class variables and bound plural variables. PRC*, then, is an extension of PV's PRC, as in PRC* formulæ containing bound plural variables are allowed on the right-hand side of the biconditional. Neither free plural variables nor bound predicative class variables are allowed on the right-hand side of the biconditional of PRC* on pain of contradiction.[9] PV's axiom V is among P-PV's axioms too:

(V) $(\forall F)(\forall G)(\{x: Fx\}=\{x: Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx))$.

Axiom V guarantees the existence of Dedekind-infinitely many first-order individuals in the domain. This is crucial to guarantee that second-order Peano arithmetic can be interpreted in P-PV.[10]

[9] For the very same reason, bound predicative class variables are not allowed on the right-hand side formula of PRC in PV.

[10] P-PV is presented in more detail as the system *Plural Grundgesetze* (PG) in Boccuni (2010). For a proof of semantic consistency of PG and consequently P-PV, see Boccuni (2011).

## 4. Second-Order Peano Arithmetic

Second-order Peano arithmetic is easily interpretable in P-PV.[11] The *singleton* and the *unordered pair* may be defined as usual:

$\{x\} =_{def} \{y: x=y\}$;

$\{x,y\} =_{def} \{z: z=x \vee z=y\}$.

The usual Wiener-Kuratowski definition of the *ordered pair* is easily provided:

$(x,y) =_{def} \{\{x\},\{x,y\}\}$.

In $L_{P\text{-}PV}$, *natural numbers* may be defined inductively:

$0 =_{def} \{x: x \neq x\}$;
$1 =_{def} \{x: x=0\}$;
$2 =_{def} \{x: x=1\}$;

and so on. In general, the *successor* of a number is its singleton.

A plurality $X$ is said to be *inductive* whenever it contains 0 and it is closed under the successor. The usual definition of the set of natural numbers may be given in terms of pluralities. First, a predicative class **N** is defined and, secondly, the corresponding set ω is introduced:

$\mathbf{N}x =_{def} \forall Y\,(Y \text{ is inductive} \rightarrow Yx)$;

$\omega =_{def} \{x: \mathbf{N}x\}$.

The derivation of a plural formulation of second-order Peano axioms easily follows.[12]

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11 See Boccuni (2010).

12 Given the previous definitions, the following formulations of second-order Peano axioms are derivable in P-PV, where the singular variables $x$ and $y$ are restricted to ω (see Boccuni (2010) for the details):

1 $\mathbf{N}0$;

## 5. The Theort S-PV

I shall now consider an augmentation of PV with full SOL. I shall call the resulting theory *Second-Order Predicative V* (S-PV). Augment $L_{PV}$ with

(3.1) a further round of second-order variables *X, Y, Z, …* varying over classes in general, i.e. over the domain of all the classes of the first-order individuals of S-PV. I will call these classes *general*.

The axioms of this theory include an axiom of impredicative comprehension

(PIC) $\exists X \, \forall x \, (Xx \leftrightarrow \varphi)$

where $\varphi$ does not contain *X* free; a predicative comprehension axiom

($PRC^+$) $\exists F \forall x \, (Fx \leftrightarrow \psi)$

where $\psi$ does not contain *F* free, nor bound predicative class variables, nor general class variables neither free nor bound; and axiom V from PV

(V) $(\forall F)(\forall G)(\{x: Fx\}=\{x: Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx))$.

Unlike PRC*, $PRC^+$ is *not* the result of extending the formulæ allowed on the right-hand side of PV's PRC. The formulæ permitted on the right-hand side of $PRC^+$, in fact, cannot contain S-PV's general class variables *X, Y, Z, …*, *neither free nor bound*. These restrictions are needed on pain of contradiction. In fact, as general class variables vary over all classes of first-order individuals, they also vary over predicative classes: allowing bound general class variables in the right-hand side formula of $PRC^+$ would amount to allow bound predicative class variables, i.e. to define predicative classes *impredicatively*. The resulting impredicativity in $PRC^+$, along with the assumption that

2 $\forall x(\{x\} \neq 0)$;

3 $\forall x \, \exists y(y=\{x\})$;

4 $\forall x \forall y(\{x\}=\{y\} \rightarrow x=y)$;

5 $\forall X \, (X0 \wedge \forall x \, (Xx \rightarrow X\{x\}) \rightarrow \forall x \, (Xx))$.

predicative classes define sets, would lead to inconsistency. Moreover, assume free general class variables were allowed in $PRC^+$: this would amount to allow each general class to define a set, which would easily lead to inconsistency. In fact, in PIC, the general class $Xx \leftrightarrow x \notin x$ may be defined, where membership may be defined as follows: $x \in y =_{def} \exists F\ (y = \{z: Fz\} \wedge Fx)$. The formula $Fx \leftrightarrow Xx$ would be a valid instance of $PRC^+$, where to $F$ there would correspond the set $\{F\}$. Now, would $\{F\}$ belong to $F$? Say it did, then, on the grounds of the definitions of $F$ and $X$, $\{F\}$ would not be a member of itself, thus it would not belong to $F$. Now, say $\{F\}$ did not belong to $F$, then, given the definition of $X$, it would not satisfy the condition $x \notin x$ and, thus, it would belong to $F$.[13]

## 6. Mathematical Expressiveness

The different applicability of PL and SOL to PV depends on the different interpretations they provide to the underlying second-order logic. S-PV operates with two sorts of second-order variables whose domains are one a sub-domain of the other: the domain of the predicative classes is a sub-set of the domain of the general classes. This requires S-PV's axiom $PRC^+$ to be restricted on pain of contradiction, in such a way that it cannot deliver interesting theorems which instead are theorems of P-PV. This leads to a significant difference in the *mathematical expressiveness* of the set-theories resulting from the two different augmentations.

S-PV, in fact, interprets second-order Peano arithmetic, as augmenting PV with SOL provides enough consistency strength to recover full second-order induction: 0 and the successor may be defined as in P-PV, and an impredicative definition of **N** may be provided as usual, as a valid instance of PIC. From this, full induction is recovered. Nevertheless, in order to avoid inconsistency, S-PV requires such restrictions that it is not able to derive desirable

[13] See Burgess (2005, § 2.3d) where a three-sorted second-order language with an impredicative second-order comprehension axiom, a predicative second-order comprehension axiom and an axiom stating that, to every predicative class, there corresponds a set, is sketched. In Burgess (2005), not all definable classes determine extensions - the ones defined impredicatively "float" over sets. This is exactly what happens in S-PV. Analogously, in P-PV not all pluralities correspond to classes and, consequently, to sets: this is implied by the restriction concerning free plural variables in PRC*. Nevertheless, in what follows I will show how P-PV is significantly different from S-PV and consequently Burgess (2005). See also Boccuni (2010) and (2011).

theorems, like the one asserting the existence of the *set* of all natural numbers. In S-PV, we are able to characterise the general class of the natural numbers, but we are not able to characterise the *set* of the natural numbers in the first-order domain.

In P-PV, on the other hand, bound plural variables are allowed on the right-hand side of PRC*, because, as plural variables and predicative class variables range over distinct domains, impredicativity is not involved in the definition of predicative classes in PRC*.[14] Consequently, in P-PV it is possible to have full (plural) comprehension interacting with axiom V.

In P-PV, unlike S-PV, the set of all natural numbers can be explicitly defined. In general, in P-PV there are sets which cannot be in S-PV. Given the formulæ permitted in PRC*, predicative classes in P-PV may be defined by $\Sigma^1_1$-formulæ of the form ∃X...X..., whereas in S-PV only $\Sigma^0_1$-formulæ are allowed to define predicative classes. Thus, there is a collection of PRC*-classes in P-PV, which cannot be defined in S-PV. P-PV provides the means to express more theorems about the universe of discourse than S-PV. In general, in P-PV we can characterise the individuals of the first-order domain - in particular, natural numbers - in a subtler, more effective way than in S-PV: in fact, we are able to define more (predicative) classes in P-PV than in S-PV and then, via axiom V, to individuate more sets in P-PV than in S-PV.[15]

In spite of their equi-interpretability, there are substantial differences between PL and SOL, which may be found in their applicability to mathematical discourse, in particular to some consistent Fregean set-theory.

Thus, even though we lack a logical criterion for distinguishing between PL and SOL, we may still distinguish among them investigating the specific theories which PL and SOL are applied to respectively. In this respect, PL has shown to have interesting potential. Whether PL offers a logically relevant alternative to SOL may be assessed by examining the derivational capacity of the formal systems it underlies to, in particular of set-theories.

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14 PRC$^+$ and PRC*, then, are both *predicative* comprehension axioms: PRC$^+$ is predicative, because quantification over predicative and general classes is not allowed in the formulæ on the right-hand side of the biconditional; PRC* is predicative, because plural quantification does not involve a domain of classes of any kind, but just the first-order individuals.

15 This result may be of some interest to any one working on principles of abstraction. In fact, it is debated what exactly the boundary is that comprehension cannot trespass, without comprehension together with Frege's Basic Law V give rise to inconsistency. See Heck (1996), Ferreira and Wehemeir (2002).

As a matter of fact, ontological arguments alone in favour or against either plural quantification or second-order quantification are not conclusive.[16] I suggest that, to this extent, it may be useful to approach the issue also from a derivational perspective. Beside the ontological accounts, the consideration of what is going to be derivationally achieved in the light of either PL or SOL may motivate and drive our preferences towards one over the other on sufficiently solid grounds.

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16 On this, see Boolos (1985).