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Fuzzy Sets and Aristotelian principles

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ABSTRACT. This paper examines the meaning of the Aristotelian principles, when Booleans are more than two. If the number of possible degrees of membership, although finished, is not a power of two, or if possible degrees of membership are all taken into a compact of R, we don't find a denial ensuring the validity of both the Not Contradiction and Excluded Middle principles. We will see, however, that if Booleans are taken by Definition 1, there is formal respect of Aristotelian Principles of Excluded Middle and Not Contradiction by the same denial. We will see that the introduction of adverbs distinct from negation and affirmation, and the introduction of conjunctions distinct from intersection and union, induces the validity of Aristotelian principles. We also will describe the inferences established when the Booleans numbers are three.

1. Definitions and Statement 1

We will use $\cap$ symbol (cap) to mean intersection and $\cup$ symbol (cup) to mean union; the complementary of a set A will be written as $\bar{A}$, and l degree of membership (or "boolean") of an element x to a set A will be explained in this way:

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$$A \coloneqq \{(x, \lambda)\} \quad \textbf{1}$$

Intersection is commutative and associative, and it's distributive on the union; 0 value absorbs; 1 value is neutral.
Union is commutative and associative, and it's distributive on intersection; 1 value absorbs; 0 value is neutral. Denial respects De Morgan's laws :

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \textbf{2}$$

$$\overline{\boldsymbol{A} \cap \boldsymbol{B}} = \overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}} \quad \textbf{3}$$

All those properties are always true, but the Statement 1 can be true or not, it depends on the truth tables. It is:

> Statement 1: a) There are two not empty sets. If their intersection is empty set, then every element does not belongs to one of two those at least. b) There are two not Universe sets. If their union is the Universe set, then every element belongs with "1" degree to one of two those at least.

We will see in this paper even truth tables that allow Statement 1 and truth tables that don't allow it.
We'll use symbol ø in order to indicate empty set and capital letter U to indicate Universe set.
We summarize now some interesting results (we don't prove them because there is no space to do it).

- If Intersection is defined on booleans by minimum's norm, then a map bringing 1 to 0 and vice versa is a denial if and only if it decreases (not strictly at least).

- If Statement 1 is true, then:

$$A \cap \overline{A} = \emptyset \Leftrightarrow \overline{\lambda} = \{1 \; if \; \lambda = 0; 0 \; otherwise$$

$$A \cup \overline{A} = \mathrm{U} \Leftrightarrow \overline{\lambda} = \{0 \; if \; \lambda = 1; 0 \; otherwise$$

If, instead, Statement 1 is false, the just introduced maps are not denials, we mean they don't respect De Morgan's Laws.

2. Statement 1 False

We state a boolean belongs to this array:
V= [0, 0.33,0.67, 1]
We state these truth tables:
Denial: $\overline{\lambda} = 1 - \lambda$
Intersection, by the following table:

$\cap$	0	0.33	0.67	1
0	0	0	0	0
0.33	0	0.33	0	0.33
0.67	0	0	0.67	0.67
1	0	0.33	0.67	1

Table 1: intersection
Union, by the following table:

$\cup$	0	0.33	0.67	1
0	0	0.33	0.67	1
0.33	0.33	0.33	1	1
0.67	0.67	1	0.67	1
1	1	1	1	1

Table 2: Union
There are four remarkable sets: empty one, Universe, *K(0.33)* whose members belongs all with Boolean 0.33, *K(0.67)* whose members belongs all with 0.67.
We say about inclusion $A \subseteq B$ there are four cases:
1) Case 0: "$A \subseteq B$" false;
2) Case 0.33: Degree of "$A \subseteq B$" is 0.33;

3) Case 0.67: Degree of “$A \subseteq B$” is 0.67;
4) Case 1: “$A \subseteq B$” true.
The following table reflects degrees of inclusions among the remarkable sets we defined.

B	Empty set	K(0.33)	K(0.67)	Universe
Empty Set $\subseteq$B	1	1	1	1
K(0.33)$\subseteq$B	0.67	1	0.67	1
K(0.67) $\subseteq$B	0.33	0.33	1	1
Universe $\subseteq$B	0	0.33	0.67	1

Table 3: inclusions

Two independent sources are given, each degree of truth reflects a different situation (depending on the source is in "1" or "0" mode).

Sources	Booleans
0 0	0
1 0	0.33
0 1	0.67
1 1	1

Table 4: sources and booleans

We can say that Statement 1 is false with these truth tables. We can also say that Aristotelian Principles are safe. We call the sources *x, y*: they can belong as elements to a set, with degrees of membership to it equal to their modes. We can formalize this discussion:

Definition 1

A family of sets let be considered: empty set and Universe set must belong to this family. If we make a set-operation on one or more of her members, the result must be one of her members.

The sets belonging to this family are taken as Booleans. So, all the set – operations we can make on an element or more elements are those we do on their Booleans.

If we use Definition 1 in order to introduce Booleans, then Aristotelian Principles are always safe. We underline not only denial, but also other adverbs exist. Let be a set *A*. Let be *K(x)*, the set whose members belong to it, all with the same boolean *x* (set taken into the family introduced in Definition 1). We can write, in order to transform *A* to Â(x):

$$\left(\bar{A} \cap K(x)\right) \cup \left(A \cap \bar{K}(x)\right) = \hat{A}(x) \qquad \mathbf{4}$$

3 Statement 1 is True

We state a Boolean can be taken into three values: 0, 0.5 and 1. We can sort them in six different ways:

Name of the sorting	First place	Second place	Third place
P1	0	0.5	1
P2	0	1	0.5
P3	1	0	0.5
P4	1	0.5	0
P5	0.5	0	1
P6	0.5	1	0

Table 5: six sortings of three elements (booleans "0","0.5","1")

There are six different maps (we will call them “main adverbs”) that bring a Boolean to another Boolean, as the following table shows.

Name of the Adverb	0 is transformed into	0.5 is transformed into	1 is transformed into
A1	0	0.5	1
A2	0	1	0.5
A3	1	0	0.5
A4	1	0.5	0
A5	0.5	0	1
A6	0.5	1	0

Table 6: six possible adverbs

If an element belongs (with two different degrees) to two different sets, we can ask what of those two degrees is more important than the other one according to one of the sorting modes shown in Table 5. We, asking that, define six different conjunctions; each of them gives to the element the Boolean "winner" according to its dual sorting.

CJ1	0	0.5	1
0	0	0	0
0.5	0	0.5	0.5
1	0	0.5	1

Table 7: truth table of "CJ1"

CJ2	0	0.5	1
0	0	0	0
0.5	0	0.5	1
1	0	1	1

Table 8: truth table of "CJ2"

CJ3	0	0.5	1
0	0	0	1
0.5	0	0.5	1
1	1	1	1

Table 9: truth table of "CJ3"

CJ4	0	0.5	1
0	0	0.5	1
0.5	0.5	0.5	1
1	1	1	1

Table 10: truth table of "CJ4"

CJ5	0	0.5	1
0	0	0.5	0
0.5	0.5	0.5	0.5

1	0	0.5	1

Table 11: truth table of "CJ5"

CJ6	0	0.5	1
0	0	0.5	1
0.5	0.5	0.5	0.5
1	1	0.5	1

Table 12: truth table of "CJ6"

You can understand Statement 1 is true (CJ1 and CJ4 are intersection and union we know).

The equations reflecting De Morgan's laws are shown now:

$A2(A \quad CJ1 \quad B) = A2(A)CJ2A2(B)$ **5 (De Morgan on CJ1)**
$A3(A \quad CJ1 \quad B) = A3(A)CJ3A3(B)$
$A4(A \quad CJ1 \quad B) = A4(A)CJ4A4(B)$
$A5(A \quad CJ1 \quad B) = A5(A)CJ5A5(B)$
$A6(A \quad CJ1 \quad B) = A6(A)CJ6A6(B)$

6 (De Morgan on CJ2)
$A2(A \quad CJ2 \quad B) = A2(A)CJ1A2(B)$
$A3(A \quad CJ2 \quad B) = A3(A)CJ4A3(B)$
$A4(A \quad CJ2 \quad B) = A4(A)CJ3A4(B)$
$A5(A \quad CJ2 \quad B) = A5(A)CJ6A5(B)$
$A6(A \quad CJ2 \quad B) = A6(A)CJ5A6(B)$

$A2(A \quad CJ3 \quad B) = A2(A)CJ5A2(B)$ **7 (De Morgan on CJ3)**
$A3(A \quad CJ3 \quad B) = A3(A)CJ6A3(B)$
$A4(A \quad CJ3 \quad B) = A4(A)CJ2A4(B)$
$A5(A \quad CJ3 \quad B) = A5(A)CJ4A5(B)$
$A6(A \quad CJ3 \quad B) = A6(A)CJ1A6(B)$
$A2(A \quad CJ4 \quad B) = A2(A)CJ6A2(B)$ **8 (De Morgan on CJ4)**
$A3(A \quad CJ4 \quad B) = A3(A)CJ5A3(B)$
$A4(A \quad CJ4 \quad B) = A4(A)CJ1A4(B)$
$A5(A \quad CJ4 \quad B) = A5(A)CJ3A5(B)$
$A6(A \quad CJ4 \quad B) = A6(A)CJ2A6(B)$
$A2(A \quad CJ5 \quad B) = A2(A)CJ3A2(B)$ **9 (De Morgan on CJ5)**
$A3(A \quad CJ5 \quad B) = A3(A)CJ2A3(B)$
$A4(A \quad CJ5 \quad B) = A4(A)CJ6A4(B)$
$A5(A \quad CJ5 \quad B) = A5(A)CJ1A5(B)$
$A6(A \quad CJ5 \quad B) = A6(A)CJ4A6(B)$
$A2(A \quad CJ6 \quad B) = A2(A)CJ4A2(B)$ **10 (De Morgan on CJ6)**

$A3(A \quad CJ6 \quad B) = A3(A) CJ1 A3(B)$
$A4(A \quad CJ6 \quad B) = A4(A) CJ5 A4(B)$
$A5(A \quad CJ6 \quad B) = A5(A) CJ2 A5(B)$
$A6(A \quad CJ6 \quad B) = A6(A) CJ3 A6(B)$

When one repeats A2, A4 and A5 two times, he has the starting set. When one repeats A3 two times, he has A6, and when one repeats two times A6 he has A3.

We can write a form of Not-contradiction and excluded-middle principles:

$(A6(A) \quad CJ2 \quad A3(A)) CJ1 \quad A = \phi$ **11**

$(A3(A) CJ3 A6(A)) CJ4 \quad A = U$ **12**

There are also other adverbs than A1…A6. They are listed in the following table.

Adverb	0	0.5	1	How we perform it:
AA1	0	0	0.5	$A \quad CJ1 \quad A3(A)$
AA2	0	0	1	$A \quad CJ1 \quad A5(A)$
AA3	0.5	0.5	0	$A5(A) \quad CJ5 \quad A4(A)$
AA4	0.5	0.5	1	$A5(A) \quad CJ5 \quad A$
AA5	1	1	0	$A6(A) \quad CJ4 \quad A4(A)$
AA6	1	1	0.5	$A2(A) \quad CJ4 \quad A4(A)$
AA7	0	0.5	0	$A \quad CJ1 \quad A6(A)$
AA8	0	1	0	$A6 \quad CJ1 \quad A2(A)$
AA9	0.5	0	0.5	$A5(A) \quad CJ5 \quad A2(A)$
AA10	0.5	1	0.5	$A3(A) \quad CJ6 \quad A6(A)$
AA11	1	0	1	$A \quad CJ3 \quad A3(A)$
AA12	1	0.5	1	$A \quad CJ4 \quad A4(A)$
AA13	0.5	0	0	$A5(A) \quad CJ1 \quad A6(A)$
AA14	1	0	0	$A4(A) \quad CJ1 \quad A3(A)$
AA15	0.5	1	1	$A5(A) \quad CJ4 \quad A6(A)$
AA16	0	1	1	$A \quad CJ3 \quad A6(A)$

AA17	0	0.5	0.5	$A \quad CJ5 \quad A3(A)$
AA18	1	0.5	0.5	$A3(A) \quad CJ5 \quad A4(A)$

Table 13: secondary adverbs

We have two sets A, B. If $ACJ1B = A$, then we say that A is first period-included in B. If $ACJ2B = A$, then we say A is second period-included in B. If $ACJ5B = A$, then we say A is third period-included in B. The conjunctions can be algebraically found. Let be Kronecker’s symbol d(x, y) the map bringing the couple (x, y) to the value 1 if x is equal to y, to the value 0 otherwise. The following equations can be written (x, y, z Booleans taken among 0, 0.5 and 1):

$$CJ1(x,y) = z$$
$$\delta(0,z) = \delta(0,x)\cdot\delta(0,y) + \delta(0,x)\cdot\big(\delta(0.5,y)+\delta(1,y)\big) + \delta(0,y)\cdot\big(\delta(0.5,x)+\delta(1,x)\big)$$
$$\delta(1,z) = \delta(1,x)\cdot\delta(1,y)$$
$$\delta(0.5,z) = 1-\delta(1,z)-\delta(0,z)$$

13: CJ1 written by Kronecker's symbol

$$CJ2(x,y) = z$$
$$\delta(0,z) = \delta(0,x)\cdot\delta(0,y) + \delta(0,x)\cdot\big(\delta(0.5,y)+\delta(1,y)\big) + \delta(0,y)\cdot\big(\delta(0.5,x)+\delta(1,x)\big)$$
$$\delta(0.5,z) = \delta(0.5,x)\cdot\delta(0.5,y)$$
$$\delta(1,z) = 1-\delta(0.5,z)-\delta(0,z)$$

14:CJ2 written by Kronecker's symbol

$$CJ3(x,y) = z$$
$$\delta(1,z) = \delta(1,x)\cdot\delta(1,y) + \delta(1,x)\cdot\big(\delta(0.5,y)+\delta(0,y)\big) + \delta(1,y)\cdot\big(\delta(0.5,x)+\delta(0,x)\big)$$
$$\delta(0.5,z) = \delta(0.5,x)\cdot\delta(0.5,y)$$
$$\delta(0,z) = 1-\delta(0.5,z)-\delta(1,z)$$

15: CJ3 written by Kronecker's symbol

$$CJ4(x,y) = z$$
$$\delta(1,z) = \delta(1,x)\cdot\delta(1,y) + \delta(1,x)\cdot\big(\delta(0.5,y)+\delta(0,y)\big) + \delta(1,y)\cdot\big(\delta(0.5,x)+\delta(0,x)\big)$$
$$\delta(0,z) = \delta(0,x)\cdot\delta(0,y)$$
$$\delta(0.5,z) = 1-\delta(0,z)-\delta(1,z)$$

16: CJ4 written by Kronecker's symbol

$$CJ5(x,y) = z$$
$$\delta(0.5,z) = \delta(0.5,x)\cdot\delta(0.5,y) + \delta(0.5,x)\cdot\big(\delta(1,y)+\delta(0,y)\big) + \delta(0.5,y)\cdot\big(\delta(1,x)+\delta(0,x)\big)$$
$$\delta(1,z) = \delta(1,x)\cdot\delta(1,y)$$

$$\delta(0,z) = 1 - \delta(0.5,z) - \delta(1,z)$$

17:CJ5 written by Kronecker's symbol

$$CJ6(x,y) = z$$

$$\delta(0.5,z) = \delta(0.5,x) \cdot \delta(0.5,y) + \delta(0.5,x) \cdot \big(\delta(1,y) + \delta(0,y)\big) + \delta(0.5,y) \cdot \big(\delta(1,x) + \delta(0,x)\big)$$

$$\delta(0,z) = \delta(0,x) \cdot \delta(0,y)$$

$$\delta(1,z) = 1 - \delta(0.5,z) - \delta(0,z)$$

18:CJ6 written by Kronecker's symbol

The Aristotelian Principles can be seen as it follows (Not Contradiction and Excluded Middle):

$$\delta(x,0) \cdot \delta(x,1) = 0$$

$$\delta(x,0) \cdot \delta(x,0.5) = 0$$

$$\delta(x,1) \cdot \delta(x,0.5) = 0$$

19: Not Contradiction written by Kronecker's symbol

$$\delta(0,x) = 1 - \delta(0.5,x) - \delta(1,x)$$

20: Excluded middle written by Kronecker's symbol

4.Conclusions

We defined the properties of conjunctions known as intersection and union, and of the denial adverb. We introduced the Statement 1, it was said that it may be true or not depending on the truth tables. We listed some remarkable results. We examined and met the validity of the Aristotelian principles when Statement 1 is false. We extended the validity of those when the Statement 1 is true.

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