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# Towards isomorphism to natural deduction: a highlighted sequent calculus

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ABSTRACT. Some restrictions on the standard sequent calculus rules are proposed, that guarantee isomorphism to natural deduction. These restrictions, concerning both left-side rules and the cut rule, leads to the definition of a system of ”highlighted” sequent calculus for minimal logic, whose main features will be described in this paper.

KEYWORDS: Sequent calculus, natural deduction, isomorphism, cut-rule.


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## 1. Introduction

The idea of a correspondence between sequent calculus and natural deduction comes already in Gentzen [1], together with the definitions of the systems. Since then, it has been the subject of many fundamental researches in Proof Theory, leading to the broadly shared conclusion that these systems are not isomorphic to each other[1]. In fact, whereas one could actually say that right-side and introduction rules intuitively refer to the same operation, so that the translation from the ones to the others is straightforward, translation from/to left-side to/from elimination rules entails some unwanted consequences that prevent one-to-one correspondence between the systems. Moreover, the origin of the failure of injectivity of the translation function may occur in the translation of the cut rule as well.

To be more precise, the analysis of the standard translation functions $\mathscr{F}$ from natural deduction to sequent calculus derivations and $\mathscr{G}$ from sequent calculus to natural deduction derivations—as first defined in Gentzen [1] and Prawitz [2]—leads to noticeable remarks, that explain the reason why a one-to-one mapping between natural deduction and standard sequent calculus derivations had always seemed impossible to establish:

(A) $\mathscr{F}$ is one-to-one, but there are standard sequent calculus derivations which are not image of any natural deduction derivation;

(B) $\mathscr{G}$ is many-to-one,

(C.1) $\mathscr{F}$ maps normal to non cut-free derivations, because elimination rules are translated to left-side rules with the help of the cut rule;

(C.2) whereas it is necessarily the case that $\mathscr{G}$ maps cut-free on normal derivations, it does not necessarily map Cuts on instances of non-normality, because composition—by means of which the cut rule is translated—of normal derivations may or may not preserve normality.

[1] See Tesconi [5] for a detailed discussion on this topic.

## 2. The system hgG of highlighted sequent calculus

A successful attempt to establish the desired one-to-one correspondence takes into account a modification of standard natural deduction—the nowadays well-known generalized elimination rules[2]. Stating that, among the connectives, disjunction shows a remarkable behaviour inasmuch as the left-side rule already corresponds exactly to the elimination rule, the elimination rule for disjunction is looked at as a model in order to formulate new elimination rules for the other connectives in the manner which is usual for disjunction (namely, with an arbitrary consequence).

In this paper, instead, the solution to the failure of one-to-one correspondence will be pursued "the other way around"—that is, in the direction of a modification of standard sequent calculus in order to make it isomorphic to natural deduction. As the remarks in the previous Section suggest, this task presents two different aspects, that is: the maintenance of the order of the inferences—on the one hand—and the correspondence between occurrences of cuts and occurrences of maximal formulae[3]—on the other hand. The new calculus will have to fulfill the requirement that the left-side rules always occur in the form which is image of the corresponding elimination rule but then it will exploit a notation that makes it possibile to distinguish between—so to speak—"different kinds of cuts" from the "inside" rather than from the "outside" of the system: this will be done by imposing that the right premiss of left-implication and the premiss of left-conjunction be initial sequents[4] and that left-side rules be always followed either by *by a cut or "something similar"*, that will be better specified in a few lines. Besides, a "highlight" will be assigned to the principal formula of every logical inference—so that it becomes possibile to recognize the last inference of a derivation simply by looking at its final sequent—in order to be able to recognize whether the cut rule may be applied or the "something similar" just mentioned.

[2] For accurate references about natural deduction with generalized elimination rules, see Tesconi [4].

[3] This does not authomatically imply a correspondence between derivations where cuts do not occur and normal derivations, but represents a good starting point for the development of a stronger condition.

[4] Left-disjunction and elimination of disjunction already correspond so there is no need for a specific treatment of this connective.

An informal discussion of the intended purposes of the main features of the highlighted calculus will be better appreciated after the formal definition of the system is given[5]. In the following, capital letters will denote formulae, for example $A$; Greek capital letters will denote multiset[6], for example $\Gamma$. Capital letters in script style will denote derivations, for example $\mathscr{D}$, and will be written as finite trees of formulae or sequents. When a generic derivation is meant, vertical dots will be used instead of letters.

## 2.1. The formal definition

The system **hgG** of highlighted sequent calculs for minimal logic is defined as follows:

INITIAL SEQUENT $\quad \Gamma, A \Rightarrow A$

LOGICAL RULES

$$\supset\text{L}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \langle A\rangle\end{array} \qquad B \Rightarrow B}{[A \supset B]_{\supset}, \Gamma \Rightarrow B} \qquad\qquad \supset\text{R}\ \frac{\begin{array}{c}\vdots\\ \Gamma, \langle A\rangle \Rightarrow \langle B\rangle\end{array}}{\Gamma \Rightarrow [A \supset B]_{\supset}}$$

$$\&\text{L}\ \frac{\Gamma, A \Rightarrow A}{[A\&B]_{\&}, \Gamma \Rightarrow A} \quad \&\text{L}\ \frac{\Gamma, B \Rightarrow B}{[A\&B]_{\&}, \Gamma \Rightarrow B} \qquad\qquad \&\text{R}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \langle A\rangle\end{array} \qquad \begin{array}{c}\vdots\\ \Delta \Rightarrow \langle B\rangle\end{array}}{\Gamma, \Delta \Rightarrow [A\&B]_{\&}}$$

$$\vee\text{L}\ \frac{\begin{array}{c}\vdots\\ \Gamma, \langle A\rangle \Rightarrow \langle C\rangle\end{array} \qquad \begin{array}{c}\vdots\\ \Delta, \langle B\rangle \Rightarrow \langle C\rangle\end{array}}{[A \vee B]_{\vee}, \Gamma, \Delta \Rightarrow C} \qquad\qquad \vee\text{R}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \langle A\rangle\end{array}}{\Gamma \Rightarrow [A \vee B]_{\vee}} \quad \vee\text{R}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \langle B\rangle\end{array}}{\Gamma \Rightarrow [A \vee B]_{\vee}}$$

As usual, the formula introduced in the conclusion of a logical rule is called *principal formula* and it receives a highlight $[]_*$ according to its principal connective $*$. The notation $\langle D\rangle$ means that the formula occurrence $D$ may or may not be highlighted.

[5] Analogous restrictions, for a first order logic sequent calculus, had been presented by von Plato in a talk held in Florence in April 2009. The content of the talk was published in von Plato [7].

[6] A multiset is a list with multiplicity but no order.

Formula occurrences in the multiset Γ may or may not be highlighted, in the respect of the condition that at most one formula occurrence per sequent may be highlighted, none for initial sequents.
CONCATENATION RULES (*c. r.* for short)

$$\mathit{Cut}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow [A]_*\end{array} \qquad \begin{array}{c}\vdots\\ [A]_*, \Delta \Rightarrow C\end{array}}{\Gamma, \Delta \Rightarrow C} \quad \mathit{Subst}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\\ [A]_*, \Delta \Rightarrow C\end{array}}{\Gamma, \Delta \Rightarrow C}$$

As usual, $[A]_*$ in the cut rule is called *cut-formula* and, analogously, $A$ and $[A]_*$ in the substitution rule are called *subst-formulae*.
CONDITION ON LEFT-SIDE RULES Every occurrence of left-side rules in a derivation must be followed by a concatenation rule such that the principal formula of the left-side rule corresponds to the cut- or subst-formula.

The deductive equivalence of **hgG** to standard sequent calculus is proved straightforwardly.
NOTATION The notation

$$\text{“}\mathit{Cut}\text{”}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow [A]_*\end{array} \qquad \begin{array}{c}\vdots\\ \langle A\rangle, \Delta \Rightarrow C\end{array}}{\Gamma, \Delta \Rightarrow C} \quad \text{“}\mathit{Subst}\text{”}\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\\ \langle A\rangle, \Delta \Rightarrow C\end{array}}{\Gamma, \Delta \Rightarrow C}$$

will represent by convention that the *Subst* or the *Cut* are performed *as soon as the conc-formula occurs highlighted in the derivation*; it may be exactly at that level—in that case, the appropriate concatenation rule will be performed exactly at that level—or upwards in the derivation—in that case, it will be performed upwards in the derivation, whilst the rest of the derivation remains unchanged.

## 2.2. An informal assessment

It is worthwhile to spend a few words on the main features of highlighted calculus, from an informal point of view:

(a) the right premiss of left-implication and the premiss of left-conjunction must be initial sequents;

(b) left-side rules must be followed by a concatenation rule, with the principal formula as concatenation formula.

(c) the principal formula of any inference receives a highlight: this ensures that the conditions of application of an inference rule provides *in themselves* a criterion of distinction between the two concatenation rules, without need for additional "external" conditions (such as the check of last inferences of premisses, for example), and guarantee that those rules "behave" according to the intended purpose of the calculus—this is stressed in the following points;

(d) only when the formula is highlighted in both premisses is the cut rule allowed, cases of the formula being highlighted only in the right premiss require what has been mentioned as "something similar" in the previous paragraph—the Subst rule—this ensures the correspondence between occurrences of cuts and occurrences of maximal formulae;

(e) concatenation rules cannot be applied on a formula which is not highlighted in the right premiss: this ensures that no other occurrences of concatenation rules appear in a derivation than those following a left-side rule.

It will be clear in Section 3 that points (a), (b) and (e) make up for the loss of injectivity due to permutations of rules one over another and of the cut rule over the last inference of left or right premiss, respectively—as referred to in remark (B) of Section 1—point (d) is necessary, though not sufficient, to adjust the gaps in the correspondence between cut-free and normal derivations—as referred to in remarks (C.1) and (C.2) of Section 1 .

## 3. The isomorphism between highlighted sequent calculus and natural deduction

The isomorphism between **hgG** and natural deduction is established by means of translation functions that do not give rise to the phenomena remarked in Section 1 . A detailed definition will be given for implication only, since the generalization to the other connectives is straightforward.

For an exhaustive treatment af all connectives, together with detailed examples and proofs of the results mentioned, see Tesconi [6].

### 3.1. Translation functions between highlighted sequent calculus and natural deduction

The translation functions $\mathscr{L}$ and $\mathscr{L}^{-1}$ which are going to be presented behave rather much like $\mathscr{G}$ and $\mathscr{F}$, as far as initial sequents/assumptions and right-side/introduction rules are concerned; the main differences obviously regard the relationship between left-side and elimination rules, which affects as a consequence the relationship between the concatenation rules and concatenation of derivations.

Axioms and assumptions:

The image under $\mathscr{L}$ of the axiom $\Gamma, A \Rightarrow A$ are the open assumptions $[\Gamma, A]$; the image under $\mathscr{L}^{-1}$ of the open assumption $[A]$ is the axiom $A \Rightarrow A$

From introduction to right-side rules:

- the introduction of implication

$$\supset\mathrm{I}_x \dfrac{\begin{array}{c}[A]_x, \Gamma \\ \mathscr{D} \\ B\end{array}}{A \supset B}$$

  corresponds to the right implication

$$\supset\mathrm{R} \dfrac{\begin{array}{c}\mathscr{L}^{-1}(\mathscr{D}) \\ \langle A \rangle, \Gamma \Rightarrow \langle B \rangle\end{array}}{\Gamma \Rightarrow [A \supset B]_{\supset}}$$

- the introduction of conjunction corresponds to the right conjunction;
- the introduction of disjunction corresponds to the right disjunction.

From right-side to introduction rules:

- the right implication

$$\supset\mathrm{R} \dfrac{\begin{array}{c}\mathscr{D} \\ \langle A \rangle, \Gamma \Rightarrow \langle B \rangle\end{array}}{\Gamma \Rightarrow [A \supset B]_{\supset}}$$

corresponds to the introduction of implication

$$\supset\mathrm{I}_x \dfrac{\begin{matrix}[A]_x, \Gamma \\ \mathscr{L}(\mathscr{D}) \\ B\end{matrix}}{A \supset B}$$

- the right conjunction corresponds to the introduction of conjunction;

- the right disjunction corresponds to the introduction of disjunction.

It is worthwhile to remark that, as a consequence of the given definition, the introduction rule for a connective $*$ is the only rule whose translation in **hgG** has the labelled formula $[D]_*$ in the succedent.
<u>From elimination rules to left-side rules:</u>

- the elimination of implication

$$\supset\mathrm{E} \dfrac{\begin{matrix}\Gamma \\ \mathscr{D} \\ A \supset B\end{matrix} \qquad \begin{matrix}\Delta \\ \mathscr{P} \\ A\end{matrix}}{B}$$

  corresponds to one of the following derivations, according to how the major premiss $A \supset B$ is inferred:

  – if it is conclusion of an introduction of implication

$$\supset\mathrm{I}_x \dfrac{\begin{matrix}[A]_x, \Gamma \\ \mathscr{D}_1 \\ B\end{matrix}}{A \supset B}$$

  then the translation is

$$Cut \dfrac{\supset\mathrm{R} \dfrac{\begin{matrix}\mathscr{L}^{-1}(\mathscr{D}_1) \\ \Gamma, \langle A \rangle \Rightarrow \langle B \rangle\end{matrix}}{\Gamma \Rightarrow [A \supset B]_\supset} \qquad \dfrac{\begin{matrix}\mathscr{L}^{-1}(\mathscr{P}) \\ \Delta \Rightarrow \langle A \rangle\end{matrix} \qquad B \Rightarrow B}{\Delta, [A \supset B]_\supset \Rightarrow B} \supset\mathrm{L}}{\Gamma, \Delta \Rightarrow B}$$

– it is

$$
\textit{Subst}\ \dfrac{\begin{array}{c}\mathscr{L}^{-1}(\mathscr{D})\\ \Gamma \Rightarrow A \supset B\end{array} \qquad \dfrac{\begin{array}{c}\mathscr{L}^{-1}(\mathscr{P})\\ \Delta \Rightarrow \langle A\rangle\end{array} \qquad B \Rightarrow B}{\Delta, [A \supset B]_{\supset} \Rightarrow B}\ \supset\mathrm{L}}{\Gamma, \Delta \Rightarrow B}
$$

otherwhise;

- analogously for elimination of conjunction and elimination of disjunction.

Now, the opposite direction requires more attention, because left-side rules never come alone in a highlighted sequent calculus derivation, so a translation that does not take into account this condition would not be representative. Instead, we choose to translate the left-side rule together with the suitable concatenation rule that follows, explicitly developing all the possible combinations[7].
From left-side rules to elimination rules:

- the left implication followed by a Cut

$$
\textit{Cut}\ \dfrac{\supset\mathrm{R}\ \dfrac{\begin{array}{c}\mathscr{D}_1\\ \Delta, \langle A\rangle \Rightarrow \langle B\rangle\end{array}}{\Delta \Rightarrow [A \supset B]_{\supset}} \qquad \dfrac{\begin{array}{c}\mathscr{D}_2\\ \Gamma \Rightarrow \langle A\rangle\end{array} \qquad B \Rightarrow B}{\Gamma, [A \supset B]_{\supset} \Rightarrow B}\ \supset\mathrm{L}}{\Delta, \Gamma \Rightarrow B}
$$

corresponds to the following derivation

$$
\supset\mathrm{E}\ \dfrac{\supset\mathrm{I}_x\ \dfrac{\begin{array}{c}\Delta, [A]_x\\ \mathscr{L}(\mathscr{D}_1)\\ B\end{array}}{A \supset B} \qquad \begin{array}{c}\Gamma\\ \mathscr{L}(\mathscr{D}_2)\\ A\end{array}}{B}
$$

[7] Thus, the problem is avoided—which cannot be avoided in the standard system—that the translation of an instance of the cut rule with an axiom as its left premiss is the same as the translation of the derivation of its right premiss.

- the left implication followed by a Subst

$$
\dfrac{\begin{array}{c}\mathscr{D}_1\\ \Delta \Rightarrow A \supset B\end{array} \qquad \dfrac{\begin{array}{c}\mathscr{D}_2\\ \Gamma \Rightarrow \langle A \rangle\end{array} \qquad B \Rightarrow B}{\Gamma, [A \supset B]_{\supset} \Rightarrow B}\supset L}{\Delta, \Gamma \Rightarrow B} Subst
$$

  corresponds to the elimination of implication

$$
\supset E\ \dfrac{\begin{array}{c}\Delta\\ \mathscr{L}(\mathscr{D}_1)\\ A \supset B\end{array} \qquad \begin{array}{c}\Gamma\\ \mathscr{L}(\mathscr{D}_2)\\ A\end{array}}{B}
$$

  where the major premiss is not inferred by introduction of implication;

- analogously for left conjunction and left disjunction.

As already stressed, no other occurrences of the concatenation rules may appear in a derivation other than those following a left-side rule, because the concatenation formula in the right premiss must be highlighted. Thus, at the same time, the definition of translation for left-side rules provides a formal definition for the translation of concatenation rules, which intuitively corresponds to ”plugging” derivations together—a procedure that do not correspond to any formal operation of natural deduction. That means that the role of concatenation rules should better be seen in the context of the correspondence between left-side and elimination rules rather than on their own—actually, it could be said that the correspondence actually holds between elimination and left-side plus concatenation rules[8].

### 3.2. What guarantees isomorphism: the order of inferences

The analysis of the above definitions, compared to those of $\mathscr{F}$ and $\mathscr{G}$, shows that the remarks listed in Section1 are not valid for $\mathscr{L}$ and $\mathscr{L}^{-1}$.

As a first thing, the definition of **hgG** makes sure that all the highlighted sequent calculus derivations are images of natural deduction derivations.

[8] See also Tesconi [5].

Secondly, $\mathscr{L}$ is one-to-one as well as $\mathscr{L}^{-1}$. Permutations of rules one over another are forbidden in the highlighted sequent calculus by the peculiar structure of left-side rules themselves, recalled in point (a) of Section 2.2 . For example, of the structures of the following standard sequent calculus derivations

$$Cut\ \dfrac{\begin{matrix}\vdots\\ \Sigma \Rightarrow B\supset C\end{matrix} \qquad {\supset}\mathrm{L}\ \dfrac{Cut\ \dfrac{\begin{matrix}\vdots\\ \Delta\Rightarrow A\supset B\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\\ \Gamma\Rightarrow A\end{matrix}\qquad B\Rightarrow B}{\Gamma, A\supset B\Rightarrow B}\ {\supset}\mathrm{L}}{\Gamma,\Delta\Rightarrow B} \qquad C\Rightarrow C}{\Gamma,\Delta,B\supset C\Rightarrow C}}{\Gamma,\Delta,\Sigma\Rightarrow C}$$

and

$$Cut\ \dfrac{\begin{matrix}\vdots\\ \Delta\Rightarrow A\supset B\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\\ \Gamma\Rightarrow A\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\\ \Sigma\Rightarrow B\supset C\end{matrix}\qquad \dfrac{B\Rightarrow B\qquad C\Rightarrow C}{B, B\supset C\Rightarrow C}\ {\supset}\mathrm{L}}{\Sigma, B\Rightarrow C}\ Cut}{\Gamma, A\supset B,\Delta\Rightarrow C}\ {\supset}\mathrm{L}}{\Gamma,\Delta,\Sigma\Rightarrow C}$$

that share a unique image in natural deduction

$${\supset}\mathrm{E}\ \dfrac{{\supset}\mathrm{E}\ \dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad \begin{matrix}\Delta\\ \vdots\\ A\supset B\end{matrix}}{B}\qquad \begin{matrix}\Sigma\\ \vdots\\ B\supset C\end{matrix}}{C}$$

only the first one can be that of a highlighted sequent calculus derivation, because in the other one—obtained by permutation of a left-rule over another—the right premiss of the lower left-side rule is not an axiom.

Moreover, permutations of left-rules over a Cut are forbidden by the condition on left-rules recalled in point (b) in Section 2.2 . For example, the structure of the standard sequent calculus derivation

$$
\mathit{Cut}\;\dfrac{\begin{matrix}\vdots\\ \Sigma\Rightarrow B\supset C\end{matrix} \qquad \mathit{Cut}\;\dfrac{\begin{matrix}\vdots\\ \Delta\Rightarrow A\supset B\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma\Rightarrow A\end{matrix}\qquad B\Rightarrow B}{\Gamma, A\supset B\Rightarrow B}\,{\supset}\mathrm{L} \qquad C\Rightarrow C}{\Gamma, A\supset B, B\supset C\Rightarrow C}\,{\supset}\mathrm{L}}{\Gamma,\Delta,B\supset C\Rightarrow C}}{\Gamma,\Delta,\Sigma\Rightarrow C}
$$

that results from the example by permutation of left-rule over a cut, and share the same image in natural deduction, cannot again be that of a highlighted sequent calculus because the upper left-implication is not followed by a concatenation rule on its principal formula.

Finally, permutations of Cuts over the last inference of their left premiss are forbidden by the condition on left-rules recalled in point (b) in Section 2.2 and permutations of Cuts over the last inference of their right premiss are forbidden by the requirement in the definition of concatenation rules that the concatenation formula be highlighted in their right premiss, as recalled in point (e) of Section 2.2 .

### 3.3. What guarantees isomorphism: the correspondence between cut-free and normal derivations

The very definition of the translation functions shows that instances of the cut rule and instances of maximal formulaes fully corresponds, that is: $A$ is a maximal formula in $\mathscr{D}$ if and only if $[A]_*$ is a cut-formula in $\mathscr{L}^{-1}(\mathscr{D})$ and $[A]_*$ is a cut-formula in $\mathscr{D}$ if and only if $A$ is a maximal formula in $\mathscr{L}(\mathscr{D})$, as anticipated in point (d) of Section 2.2 .

However, when the disjunction is included in the system, this does not amount to establish a full correspondence between derivations where no occurrences of the cut rule occur—let us call them *cut-less* derivations—and normal derivations. In fact, it may still happen that a cut-less derivation corresponds to a derivation in which maximal fragments appear—that is, pairs of introduction and elimination rules that would produce a maximal formula, if occurring consecutively, are instead separated by finitely many applications of elimination of disjunction. These cases are treated in natural deduction by means of permutations that bring such inferences next to each other in order to let the maximal formula come to light and finally being able to remove it by means of the appropriate reduction. The same happens in the highlighted sequent calculus. In a cut-less derivation, it may still happen that the left premiss of a Subst is conclusion of a left disjunction (plus Subst); then the rules has to be permuted one over the other—repeatedly, if necessary—in the following way[9]:

PERMUTATIVE CONVERSIONS

$$\dfrac{Subst\;\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A \vee B\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \langle A\rangle, \Delta \Rightarrow \langle C\rangle\end{matrix} \qquad \begin{matrix}\vdots\\ \langle B\rangle, \Sigma \Rightarrow \langle C\rangle\end{matrix}}{[A \vee B]_{\vee}, \Gamma, \Sigma \Rightarrow C}\,\vee\mathrm{L}}{\Gamma, \Delta, \Sigma \Rightarrow C} \qquad \begin{matrix}\vdots\\ [C]_{*}, \Phi \Rightarrow D\end{matrix}}{\Gamma, \Delta, \Sigma, \Phi \Rightarrow D}\,Subst$$

becomes

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A \vee B\end{matrix} \qquad \dfrac{c.\ r.\;\dfrac{\begin{matrix}\vdots\\ \langle A\rangle, \Delta \Rightarrow \langle C\rangle\end{matrix} \quad \begin{matrix}\vdots\\ [C]_{*}, \Phi \Rightarrow D\end{matrix}}{A, \Phi, \Delta \Rightarrow D} \qquad \dfrac{\begin{matrix}\vdots\\ \langle B\rangle, \Sigma \Rightarrow \langle C\rangle\end{matrix} \quad \begin{matrix}\vdots\\ [C]_{*}, \Phi \Rightarrow D\end{matrix}}{B, \Phi, \Sigma \Rightarrow D}\;c.\ r.}{[A \vee B]_{\vee}, \Phi, \Delta, \Sigma \Rightarrow D}\,\vee\mathrm{L}}{\Gamma, \Delta, \Sigma, \Phi \Rightarrow D}\,Subst$$

Thus, a highlighted sequent calculus derivation $\mathscr{D}$ will be more correctly said to be *cut-free* if and only if it is cut-less and the left premisses of the occurrences of Subst in $\mathscr{D}$ are either axioms or conclusions of left implication/conjunction plus Subst. It goes without saying that a cut-free derivation is necessarily cut-less but the viceversa does not hold. Now, correspondence is finally obtained between cut-free and normal derivations.

Admissibility of the cut rule is trivially expected, thanks to isomorphism to natural deduction (though, obviously, the steps are slightly different from those for the standard system); as a consequence, admissibility of the subst rule, instead, does not hold—this is also expected, since the presence of concatenation rules is necessary to fulfill the condition on left-side rules in **hgG**. However, a Cut Elimination Theorem holds for **hgG**, which leads to the Subformula Property.

[9] It may be assumed without loss of generality that the first derivation is cut-less.

### 3.4. Conclusive remarks

In view of the formal results presented so far, it is worthwhile to briefly outline the sketch of a comparison between composition of derivation in natural deduction, the standard cut rule and the concatenation rules in highlighted calculus.

Composition of derivations in natural deduction should be usually considered a notational device that ”stands for” the plugging of the conclusion of a certain derivation into the open assumptions of another given derivation—no matter their real proof structure. In other words, there is nothing like ”a general operation of composition” among the rules of natural deduction. Actually, composition is only needed in the procedure of normalization, when removing maximal formulaes.

Quite differently, the cut rule is a (admissible) rule of standard sequent calculus and its application is not more significant in a certain proof structure rather than in another. It may express all kinds of concatenations of derivations, not only those involving maximal formulaes.

Finally, concatenation rules in highlighted sequent calculus are—like the standard cut rule—rules of the system however—unlike standard cut rule—cannot express all kinds of concatenations but only those needed to establish a correspondence between elimination and left-side rules (this is consistent with its isomorphism to natural deduction and the already mentioned lack of a general operation of composition of this latter). In order to guarantee that no other compositions gets involved in the translation from highlighted sequent calculus to natural deduction, the conc-formula has to be highlighted in the right premise—that is, it has to correspond to the major premise of an elimination rule.

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