L&PS – Logic and Philosophy of Science
Vol. IX, No. 1, 2011, pp. 17-31

Towards a New Logic of Indicative Conditionals

Alberto Mura
Università degli Studi di Sassari
Dipartimento di Storia, Scienze dell'Uomo e della Formazione
e-mail: ammura@uniss.it

1. Background
2. The Semantics of Hypervaluations
3. Logical equivalence
4. Tri-events are logically equivalent to simple conditionals
5. Probability
6. Logical consequence
7. Philosophical impact of the present research

ABSTRACT. In this paper I will propose a refinement of the semantics of hypervaluations (Mura 2009), one in which a hypervaluation is built up on the basis of a set of valuations, instead of a single valuation. I shall define validity with respect to all the subsets of valuations. Focusing our attention on the set of valid sentences, it may easily shown that the rule substitution is restored and we may use valid schemas to represent classes of valid sentences sharing the same logical form. However, the resulting semantical theory TH turns out to be throughout a modal three-valued theory (modal symbols being definable in terms of the non modal connectives) and a fragment of it may be considered as a three-valued version of S5 system. Moreover, TH may be embedded in S5, in the sense that for every formula ϕ of TH there is a corresponding formula ϕ' of S5 such that ϕ' is S5-valid iff ϕ is TH-valid. The fundamental property of this system is that it allows the definition of a purely semantical relation of logical consequence which is coextensive to Adams' *p*-entailment with respect to simple conditional sentences, without being defined in probabilistic terms. However, probability may be well be defined on the lattice of hypervaluated tri-events, and it may be

Published by L&PS – Logic and Philosophy of Science
http://www2.units.it/episteme — ISSN: 1826-1043

proved that Adam's *p*-entailment, once extended to all tri-events, coincides with our notion of logical consequence as defined in purely semantical terms.

1. Background

According to the so called *Lewis Triviality Results* (1976) there is no suitable binary propositional connective (say '>') for representing a conditional connective (let alone a truth-functional one) that satisfies the equation $\mathbf{P}(x > y) = \mathbf{P}(y \mid x)$, where $\mathbf{P}$ is any probability function such that $\mathbf{P}(x)$ is greater than 0. On the other hand, as suggested by F. P. Ramsey in 1929 (Ramsey ([1929] 1990) and later widely endorsed, the acceptability of an indicative conditional $x > y$ grows monotonically with $\mathbf{P}(y \mid x)$, i.e. with the conditional epistemic probability of its consequent given its antecedent. Now, these two propositions, taken together, strongly support the tenet that indicative conditionals lack truth-conditions, so that $\mathbf{P}(y \mid x)$ provides a measure of the assertibility of $x > y$ but it does not represent a measure of its probability of being true. This view has been developed by E. Adams in his book *The Logic of Conditionals* (1975), where an extension of classical propositional logic and proof theory, equipped with a conditional connective '>' covering simple conditionals (i.e. conditionals of the form $x > y$ where both x and y are such that '>' does not occur in them) is developed in detail. Adams logic turns out to be in good accordance with common intuitions. Many authors (like A. Gibbard (1981), A. Appiah (1985), D. Edgington (1986), and J. Bennett (1988)), strongly influenced by Adams' work, embraced the philosophical view according to which indicative conditionals always lack truth-conditions, in spite of their assertibility.

Lewis Triviality Results depend throughout on the assumption that the meaning of conditional declarative sentences are two-valued propositions. An alternative standpoint was developed (and for a long period of time neglected) by de Finetti in 1935 (de Finetti [1936] 1995). According to de Finetti, indicative conditionals (called by him "tri-events") lack a truth-value only when the antecedent is not true and may be true or false otherwise. This view may be modeled by a three-valued logic, that de Finetti outlined in his paper, anticipating ideas of Kleene, Bochvar, and Blamey. De Finetti shows that the enlarged logico-algebraic environment of tri-events is a lattice and that every probability function **P** defined on a Boolean algebra may be uniquely extended to the generated lattice of tri-events in such a way that the equation $\mathbf{P}(x > y) = \mathbf{P}(y \mid x)$ is always satisfied provided $\mathbf{P}(x)$ is greater than 0. De Finetti's ideas were rediscovered in the AI field after a long research program that begun by G. Schay (1968) and culminated by the publication of the so called GNW theory by I. R. Goodman, H. T. Nguyen and E. A. Walker (1991). GNW theory actually is nothing but a more detailed account of de Finetti's theory of tri-events.

Unfortunately, the de Finetti-GNW theory had little impact on the philosophical debate about indicative conditionals. Among the reasons that caused this situation there is surely its inadequacy to provide a good solution of the philosophical problems concerning indicative conditionals. In particular (a) by a result due to Van McGee (1981) every many-valued logic equipped with a standard truth-functional semantics (like de Finetti-GNW theory) is at odds with Adams Proof theory of simple conditionals, and (b) there are many instances in which the de Finetti-GNW theory leads to counterintuitive results (Edgington, 2006).

2. The Semantics of Hypervaluations

In Mura (2009, pp. 223-24), the de Finetti-GNW approach has been equipped with a new semantics, called the *semantics of hypervaluations* . The resulting theory will be called here 'theory of hypervaluated tri-events'.

The typical counterintuitive instances that undermine the original de Finetti-GNW theory disappear in the theory of hypervaluated tri-events. Moreover, McGee result does not apply to this kind of non-standard semantics. This potentially opens the door to a reformulation of the de Finetti-GNW theory in such a manner that it no longer will continue to be in contrast with Adams account. We shall see that this is the case.

Unfortunately, in this semantics a key feature of formal logic appears to be lost. Valid sentences should be such only in virtue of the logical constants occurring in them. The rule of substitution, by which in a valid compound sentence every sub sentence may be replaced, *salva validitate*, with any sentence whatsoever, does not hold. For example, if p is an atomic sentence, $p \vee \neg p$ is a tautology according to the semantics of hypervaluations presented in Mura

2009, while in general sentences of the form ($\varphi \vee \neg\varphi$) are not. This prevents the use of schemas to represent classes of valid formulas.

In this paper I will propose a refinement of the semantics of hypervaluations, one in which a hypervaluation is built up on the basis of a *set* of valuations, instead of a single valuation. I shall define *validity* with respect to all the subsets of valuations. Focusing our attention on the set of valid sentences, it may easily shown that the rule substitution is restored and we may use valid schemas to represent classes of valid sentences. However, the resulting semantical theory TH turns out to be throughout a *modal three-valued theory* (modal symbols being definable in terms of the connectives) and a fragment of it may be considered as a three-valued version of S5 system. Moreover, TH may be embedded in S5 in the sense that for every formula φ_T of *T* there is a corresponding formula φ_{S5} of S5 such that φ_{S5} is S5-valid iff φ_T is T-valid.

The fundamental property of this system is that it allows the definition of a purely semantical relation of logical consequence which is coextensive to Adams' *p*-entailment with respect to ordinary sentences, without being defined in probabilistic terms. However, probability may be well be defined on the lattice of hypervaluated tri-events, and it may be proved that Adam's *p*-entailment, once extended to all tri-events, coincides with our notion of logical consequence as defined in semantical terms.

Another important issue concerns the so-called simple conditionals. These are sentences of the form C > A where neither C nor A contain occurrences of the conditional '>'. Adams restricts the syntax of his logical language so that '>' may occur only in simple conditionals. This move rules out compound of conditionals from his logic. In the present theory '>' has a standard formation rule ("if φ and ψ are sentences $\varphi > \psi$ is a sentence") so that also compound of conditionals are allowed. However, it turns out that every tri-event is truth-conditionally (i.e. logically) equivalent to a simple conditional of the form A > C, where A and C may contain occurrences of modal symbols, but necessarily they are either true or false. In the following, instead of writing '$\varphi > \psi$' we follow de Finetti in using the notation '$\psi \mid \varphi$', in the light of the result according to which conditional probability may be viewed as the probability of a conditional. The symbol '|' is here a connective at the object language level, equipped with truth-conditions (not the common metalinguistic symbol). Probability, defined over tri-events, is a function of a single variable.

The notion of hypervaluation

Let be $\mathcal{L}$ be a sentential language with a denumerable set of atomic sentences equipped with the 0-ary connectives '$\top$', '$\bot$', $\natural$, unary connectives '$\neg$', '$\uparrow$', '$\updownarrow$' and the binary connectives '$\vee$', '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', '$|$'.

Definition 1. Let $\mathcal{S}$ the set of the sentences of $\mathcal{L}$, v a valuation and V a set of valuations such that $v \in V$. The *hypervaluation associated with* v and V is the function $h_v^V : S \longrightarrow$ {t,u,f} recursively defined by the following conditions:

1. For every atomic sentence ϕ, $h_v^V(\phi) = v(\phi)$.
2. If $\phi = \neg\psi$ then
   a. $h_v^V(\phi) = \text{t}$ if $h_v^V(\psi) = \text{f}$;
   b. $h_v^V(\phi) = \text{f}$ if $h_v^V(\psi) = \text{t}$;
   c. $h_v^V(\phi) = \text{u}$ otherwise.
3. If $\phi = (\chi \vee \psi)$ then
   a. $h_v^V(\phi) = \text{t}$ if at least one of the following conditions are satisfied:
      i. $h_v^V(\chi) = \text{t}$;
      ii. $h_v^V(\psi) = \text{t}$;
      iii. All the following conditions are satisfied:
         01. for no valuation $w \in V$ $h_w^V(\chi) = \text{f}$ and $h_w^V(\psi) = \text{f}$;
         02. there exist a valuation $w' \in V$ such that either $h_{w'}^V(\chi) = \text{t}$ or $h_{w'}^V(\psi) = \text{t}$.
   b. $h_v(\phi) = \text{f}$ if at least one of the following conditions are satisfied:
      i. $h_v^V(\chi) = \text{f}$ and $h_v^V(\psi) = \text{f}$;
      ii. All the following conditions are satisfied:
         01. for every valuation $w \in V$ $h_w^V(\chi) \in$ {u,f} and $h_w^V(\psi) \in$ {u,f};
         02. there exist a valuation $w' \in V$ such that $h_{w'}^V(\chi) = \text{f}$ and $h_{w'}^V(\psi) = \text{f}$
   c. $h_v^V(\phi) = \text{u}$ otherwise.
4. If $\phi = (\chi \wedge \psi)$ then
   a. $h_v^V(\phi) = \text{t}$ if at least one of the following conditions are satisfied:
      i. $h_v^V(\chi) = \text{t}$ and $h_v^V(\psi) = \text{t}$;

ii. All the following conditions are satisfied:
   01. for every valuation $w \in V$ $h_w^V(\chi) \in \{\text{t,u}\}$ and $h_w^V(\psi) \in \{\text{t,u}\}$;
   02. there exist a valuation $w' \in V$ such that $h_{w'}^V(\chi) = \text{t}$ and $h_{w'}^V(\psi) = \text{t}$.

b. $h_v^V(\phi) = \text{f}$ if at least one of the following conditions are satisfied:
   i. Either $h_v^V(\chi) = \text{f}$ or $h_v^V(\psi) = \text{f}$;
   ii. All the following conditions are satisfied:
      01. for no valuation $w \in V$ $h_w^V(\chi) = \text{t}$ and $h_w^V(\psi) = \text{t}$;
      02. there exist a valuation $w' \in V$ such that either $h_{w'}^V(\chi) = \text{f}$ or $h_{w'}^V(\psi) = \text{f}$

c. $h_v^V(\phi) = \text{u}$ otherwise.

5. If $\phi = (\chi \mid \psi)$ then
   a. $h_v(\phi) = \text{t}$ if at least one of the following conditions are satisfied:
      i. $h_v(\chi) = \text{t}$ and $h_v(\psi) = \text{t}$;
      ii. All the following conditions are satisfied:
         01. every valuation w such that $h_w(\psi) = \text{t}$, $h_w(\chi) \in \{\text{t,u}\}$
         02. there is a valuation w' such that $h_w(\chi) = \text{t}$ and $h_w(\psi) = \text{t}$
   b. $h_v(\phi) = \text{f}$ if at least one of the following conditions are satisfied:
      i. $h_v(\chi) = \text{f}$ and $h_v(\psi) = \text{t}$;
      ii. All the following conditions are satisfied:
         01. every valuation w such that $h_w(\psi) = \text{t}$, $h_w(\chi) \in \{\text{f,u}\}$
         02. there is a valuation w' such that $h_w(\chi) = \text{f}$ and $h_w(\psi) = \text{t}$
   c. $h_v(\phi) = \text{u}$ otherwise.

6. If $\phi = (\chi \rightarrow \psi)$ then
   a. $h_v^V(\phi) = \text{t}$ if at least one of the following conditions are satisfied:
      i. $h_v^V(\psi) = \text{t}$
      ii. $h_v^V(\chi) = \text{f}$
      iii. $h_v^V(\chi) = \text{u}$ and $h_v^V(\psi) \in \{\text{t,u}\}$
   b. $h_v^V(\phi) = \text{f}$ otherwise.

7. If $\phi = (\chi \leftrightarrow \psi)$ then
   a. $h_v^V(\phi) = \text{t}$ if $h_v^V(\chi) = \text{t}$ and $h_v^V(\psi) = \text{t}$

b. $h_v^V(\phi) = \text{u}$ if $h_v^V(\chi) = \text{u}$ and $h_v^V(\psi) = \text{u}$
c. $h_v^V(\phi) = \text{f}$ if $h_v^V(\chi) = \text{f}$ and $h_v^V(\psi) = \text{f}$

8. If $\phi = \uparrow \psi$ then
   a. $h_v^V(\phi) = \text{t}$ if $h_v^V(\psi) = \text{t}$
   b. $h_v^V(\phi) = \text{f}$ otherwise.
9. If $\phi = \updownarrow \psi$ then
   a. $h_v^V(\phi) = \text{t}$ if either $h_v^V(\psi) = \text{t}$ or $h_v^V(\psi) = \text{f}$;
   b. $h_v^V(\phi) = \text{f}$ otherwise.

10. If $\phi = \top$ then $h_v^V(\phi) = \text{t}$
11. If $\phi = \bot$ then $h_v^V(\phi) = \text{f}$
12. If $\phi = \natural$ then $h_v^V(\phi) = \text{u}$.

Notice that the adopted material implication (like the unary connectives $\uparrow$ and $\updownarrow$) has a two-valued output. Due to this, the semantics of these connectives is truth-functional. The reason for the choice of the adopted material implication is due to the property by which $(\varphi \to \psi)$ is valid iff ψ is a logical consequence (in the sense to be defined) of φ as generally required for material implication.

Validity

By means of definition 1 we may define the notion of valid sentence. As we shall see it has an essential modal character.

Definition 2. φ is a valid sentence iff for every nonempty set V of valuations, φ is a valid sentence with respect to V.

Being invariant under substitution of sub sentences, validity makes room for the idea of *valid schemas*.

Modal operators

As we have anticipated, modal symbols may be defined by means of the connectives under the semantics of hypervaluations. There are two kinds of modal operators that may be defined: (a) those modal operators that, if applied to a sentence, give rise to a new sentence that may be either true or false or null and (b) those modal operators that, if applied to a sentence, give rise to a new sentence that is either true or false. Since all the operators of the kind (a) and (b) may be defined in terms of the non modal operators, the choice between operators of kind (a) and of kind (b) is just a matter of convenience in spelling out the modal properties of tri-events. Both kinds of operators have interesting properties. Operators of kind (a) seem to be preferable because they have the great advantage of preserving the usual mutual definability of necessity and possibility operators. On the other hand operators of kind (b) preserve the relationship between necessity and validity and between possibility and satisfiability (which is a typical feature of the S5 system). I shall call simply 'possible' and 'necessary' the three-valued notion reserving the name of 'possibly true' and 'necessarily true' to the latter. So using both kind of operators seems to be advisable, while only one class of them may be adopted (in this case I prefer to use operators of kind (a)).

Beyond necessity and possibility another modality of kind (b) is useful, expressing the idea that a sentence is *void* (i.e. necessarily neither true nor false).

Definition 3. The following modal operators are introduced by definition in terms of the primitive connectives

1. φ is void. $\boxtimes\varphi \stackrel{\text{def}}{=} \neg\uparrow(\varphi \vee \neg\varphi)$
2. φ is possibly true. $\diamond\!\!\!\cdot\,\varphi \stackrel{\text{def}}{=} (\top\to(\varphi|\varphi))$
3. φ is possible. $\Diamond\varphi \stackrel{\text{def}}{=} \diamond\!\!\!\cdot\,\varphi|\neg\boxtimes\varphi$
4. φ is necessary. $\Box\varphi \stackrel{\text{def}}{=} \neg\Diamond\neg\varphi$
5. φ is necessarily true. $\blacksquare\varphi \stackrel{\text{def}}{=} \Box\uparrow\varphi$

3. Logical equivalence

Two sentences that have the same truth-conditions are said to be *logically equivalent*. This is put in exact terms by the following definition:

Definition 4. φ and ψ are logically equivalent iff for every V and every $v \in V$ *it holds that* $h_v^V(\varphi) = h_v^V(\psi)$.

It follows from definition 1 that φ and ψ are logically equivalent iff the sentence $(\varphi \leftrightarrow \psi)$ is valid. Since the set of sentences is, according to the semantics of hypervaluations, a lattice with respect to conjunction and disjunction and the relation of logical equivalence is a congruence on that lattice (indeed replacing a sub sentence with a logically equivalent sentence yields a sentence logically equivalent to the original sentence), the quotient lattice $\mathcal{S}/\leftrightarrow$ provides what I call the *algebra* $\mathcal{A}$ *of the logic of hypervaluated tri-events*. $\mathcal{A}$ turns out to be a *non* distributive lattice in which however De Morgan laws are preserved.

4. Tri-events are logically equivalent to simple conditionals

$\mathcal{A}$ contains a sub lattice $\mathcal{B}$ which is a Boolean algebra and contains the set of the two-valued propositions. The elements of $\mathcal{B}$ are just the elements of $\mathcal{A}$ such that $\uparrow x = x$ or, alternatively, those elements of $\mathcal{A}$ such that $\updownarrow x = \top$. Since for every x it holds that $x = (\uparrow x) \mid (\updownarrow x)$ — this property, discovered by de Finetti (1935, p. 185), is preserved by the semantics of hypervaluations — every tri-events in $\mathcal{B}$ may be represented as a simple conditional of the form $C \mid A$ where both C and A are two-valued propositions belonging to $\mathcal{B}$. This result has a syntactic counterpart at a sentence level:

Theorem 1. Every sentence φ of $\mathcal{L}$ is logically equivalent to a sentence ψ of the form $\varphi' \mid \varphi''$ such that '|' does not occur neither in φ' nor in φ'' and every atomic sentence of both φ' and φ'' is immediately preceded by '↑' or by '↑¬'.

This result, in spite of the simplicity of its algebraic counterpart, requires a long and tedious inductive proof on the construction of sentences and it is here omitted. From theorem 1 easily follows that for every set of valuations V, $h_v^V(\varphi') \in \{t, f\}$ and $h_v^V(\varphi'') \in \{t, f\}$, so that both φ' and φ'' are two-valued sentential tri-events. It should be noticed, however, that modal symbols may occur in φ' and φ''.

5. Probability

As it is well known, according to semantical viewpoint, the absolute probability $\mathbf{P}(\varphi)$ of a proposition may be characterized as the expectation of its truth-value (considering the quantity 0 for false and 1 for true). Tri-events are not standard propositions, but they may be thought as simple conditionals. This allows to view the probability of the tri-events as the *expectation* of their truth-value *conditional on the hypothesis that they are either true or false*. The expectation of the conditional truth-value of a tri-events *x* is undefined if the probability that *x* is either true or false is 0. Given a set of valuations *V* and a valuation $v \in V$, the hypervaluation h_v^V may be viewed as an extreme probability function attributing probability 0 to all false tri-events, probability 1 to all true tri-events and being undefined when applied to those tri-events that are neither true nor false. In the light of these considerations, the probability $\mathbf{P}(x) = \mathbf{E}(x = 1 \mid x = 0 \text{ or } x = 1) = \frac{\mathbf{E}(x=1)}{\mathbf{E}(x=0 \text{ or } x=1)} = \mathbf{P}\frac{\uparrow x}{\updownarrow x}$ provided $\mathbf{P}(\updownarrow x) > 0$ (where '|'here is not our conditioning connective but the standard conditionalization operator used in probability theory). If we define first a probability function defined on $\mathcal{B}$, the above formula allows its extension to $\mathcal{A}$ in a way that avoids Lewis triviality results: for every tri-events expressed in its simple form C | A we have that $\mathbf{P}(\mathrm{C} \mid \mathrm{A}) = \frac{\mathbf{P}(\mathrm{A} \wedge \mathrm{C})}{\mathbf{P}(\mathrm{A})}$.

6. Logical consequence

Our aim is to provide semantical foundations to Adams logic of conditionals and to generalize it to compound of conditionals, i.e. to all tri-events.

Consider first the case of two sentences φ and ψ. My proposal is provided by the following definition:

Definition 5. ψ is a logical consequence of φ (symbolized as $\varphi \vDash \psi$) iff for every set *V* of hypervaluations (a) there is no element *v* of *V* such that $h_v^V(\varphi) = t$ and not $h_v^V(\psi) = t$ (preservation of truth) and (b) for every element *v* of *V* such that $h_v^V(\varphi) \in \{t, u\}$ also $h_v^V(\psi) \in \{t, u\}$ (preservation of non falsehood).

For further details on the justification of this definition see Mura, 2009, pp. 212-14.

It is easy to verify that the material implication '→' according to the semantic of hypervaluations is such that (φ → ψ) is a valid schema iff ψ is a logical consequence of φ. So our '→' with the semantic specified in the definition 1 is "right" as material implication for our notion of logical consequence. It may be proved that there are no other binary connectives that have this property. Unfortunately, however, we cannot generalize this fragment of the deduction theorem. The reason is that the underlying algebraic structure is a non distributive lattice and therefore it cannot be a Brouwerian lattice. So, in general, there is no way to define the notions of material implication and of logical consequence in such a way that the full form of the deduction theorem is provable.

Another property of logical consequence in classical logic is that, given a finite set of sentences $K=\{\varphi_1,\ldots,\varphi_n\}$, any sentence ψ is a logical consequence of K iff ψ is a logical consequence of $\varphi_1\wedge\ldots\wedge\varphi_n$. This property is not preserved in our logic. While for every i φ_i is a logical consequence of $\varphi_1\wedge\ldots\wedge\varphi_n$, $\varphi_1\wedge\ldots\wedge\varphi_n$ is *not* a logical consequence of K. Indeed in this logic the introduction rule for the conjunction is *not* valid. Thus $(\varphi \wedge \psi)$ is not in general *p*-entailed (in Adams sense) by $\{\varphi, \psi\}$. On the other hand, the same conclusion may be reached by purely logico-algebraic considerations.

From the pragmatic point of view, lack of the introduction rule for conjunction implies that the simultaneous assertion of two (or more) sentences is not equivalent to the assertion of their conjunction. This typically happens when two simple conditionals whose antecedent are logically incompatible are simultaneously asserted. For example, I can assert simultaneously that I will go by car if it rains and that I will go walking if it is good weather . I cannot assert the de Finetti's-Kleene conjunction 'I will go by car if it rains and I will go walking if it does not rain', interpreting 'if' as the conditioning connective, because this conjunction in our semantics is necessarily null. So the 'and' occurring in it, according to the present logic, may well be a connective *between speech acts*, but not a connective between conditional sentences (which, therefore, are not *conditional assertions*) .

This situation may appear strange, because we are accustomed to consider the simultaneous assertion of two or more propositions and the single assertion of their conjunction as equivalent. A solution would be to adopt a different three-valued connective for conjunction, one for which the conjunction rule is valid. Such a connective in fact exists and within the standard truth-functional semantics has been widely studied by several authors in the three-valued logics literature. Adams himself introduced it, albeit in a restricted form (applied only to simple conditionals) calling it "quasi-conjunction" (I shall adopt this term also for its general form). The truth-table of quasi-conjunction is the following:

Q u a s i - c o n j u n c t i o n
($\varphi \owedge \psi$)

		ψ: t	u	f
φ	t	t	t	f
	u	t	u	f
	f	f	f	f

The basic idea that makes plausible this connective from the point of view of partial logic is that the conjuncts that are neither true nor false are "neglected" in determining the truth-value of the conjunction, unless all conjuncts are null in which case the conjunction itself is null. Of course, in the semantics of hypervaluations the truth conditions provided by the above table must be modified. In any case $\owedge$ may be defined according to the semantic of hypervaluations in terms of the primitive connectives by means of the following schema (that turns out to be correct also for the standard truth-functional semantics):

$$\varphi \owedge \psi \overset{\text{def}}{=} (\neg{\uparrow}\neg\varphi \wedge \neg{\uparrow}\neg\psi) \mid ({\updownarrow}\, \varphi \vee {\updownarrow}\, \psi)$$

The introduction rule for quasi-conjunction is valid. Unfortunately, the elimination rule for quasi-conjunction is not. Indeed, it may be proved that there is no connective coinciding with standard conjunctions for two-valued sentences such that both introduction and elimination rules are in accordance with *p*-entailment in Adams sense (Cfr. Adams, 1998, p. 177).

However, quasi-conjunctions allows the general definition of the notion of logical consequence from a finite set of sentences that generalizes Adams *p*-entailment (which is confined to simple conditionals) in purely semantical terms, so challenging the view that conditionals lack truth conditions. Notice that, due to the fact that in this logic compactness fails, a definition extended to infinite sets of premises would be at odds with completeness of a system of deduction rules. So I stick to Adams choice of confining the definition of logical consequence to a finite set of premises. The definition is the following:

Definition 6. ψ is a logical consequence of $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ $(1 \leq n < \omega)$ (we shall write $\Gamma \models \psi$) iff either ψ is valid or there is a subset $\Gamma' = \{\varphi_{i_1}, \ldots, \varphi_{i_k}\}$ of Γ $(k \leq n)$ such that $\{\varphi_{i_1} \oslash \ldots \oslash \varphi_{i_k}\} \models \psi$

The following result shows that definition 6 encompasses Adams *p*-entailment and extends it to the lattice of tri-events equipped with the semantics of hypervaluations.

Theorem 2. Let $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ be a finite set of not void sentences of $\mathcal{L}$ and ψ a not void sentence of $\mathcal{L}$. The two following propositions are equivalent:

(a) for every probability function **P** defined for every element of Γ and for ψ it holds that $\mathbf{P}(1 - \psi) \leq \sum_{i=1}^{n}(1 - \mathbf{P}(\varphi_i))$;

(b) $\Gamma \models \psi$

The proof of this result goes beyond the space left for this paper. The proof will appear in Mura (2012).

7. Philosophical impact of the present research

The theory of hypervaluated tri-events shows that there is room for third view between the standpoint according to which indicative conditionals are two-valued propositions and the opposite standpoint according to which indicative conditionals always lack truth-conditions. According to this third view, indicative conditionals belong to partial logic. In typical cases they are true when both their antecedent and consequent are true, they are false when the antecedent is true and the consequent is false. In the other cases they lack a truth-value. From the pragmatic viewpoint, the assertibility of an indicative conditional is just the conditional probability of its consequent given its antecedent. Like in the original de Finetti-GNW theory, Lewis Triviality results do not apply to hypervaluated tri-events. Moreover, the semantics of hypervaluations allows to view Adams Logic as a fragment of a modal (three-valued) partial logic that seems to be helpful in dealing with compound of conditionals. Finally, the presence of the modal vocabulary enriches Adams theory in another direction, allowing the expression of a wider class of conditionals.

REFERENCES

ADAMS, E. W. (1975): *The Logic of Conditionals*, Dordrecht: D. Reidel.

ADAMS, E. W. (1998): *A Primer of Probability Logic*, Stanford: CSLI Publications.

APPIAH, A. (1985): *Assertion and Conditionals*, Cambridge: Cambridge University Press.

BENNETT, J. (1988): “Farewell to the Phlogiston Theory of Conditionals”, *Mind*, 97, pp. 509–27.

DE FINETTI. B. ([1936] 1995): “The Logic of Probability”, *Philosophical Studies*, 77, pp. 181–90.

EDGINGTON, D. (1986): “Do Conditionals have Truth Conditions?”, *Critica*, 18, pp. 3–30.

EDGINGTON, D. (2006): “Conditionals”, in Zalta, E. N. (ed.) *The Stanford Encyclopedia of Philosophy*, Stanford University.

GIBBARD, A. (1981): “Two Recent Theories of Conditionals“. In W. Harper *et al.* (eds.) *Ifs*, Dordrecht: D. Reidel.

GOODMAN I. R., H. T. NGUYEN, AND E. WALKER (1991): *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*, Amsterdam: North-Holland.

LEWIS, D. (1976): “Probabilities of Conditionals and Conditional Probability”, *Philosophical Review*, 85, pp. 297–315.

MCGEE V. (1981): “Finite Matrices and the Logic of Conditionals”, *Journal of Philosophical Logic*, 10, pp. 349–51.

MURA, A. (2009): “Probability and the Logic of de Finetti’s Trievents”, in Galavotti, M. C. (ed.) *Bruno de Finetti Radical Probabilist*, London: College Publications, pp. 201–42.

MURA, A. (2012): "A Partial Modal Semantics for the Adams Logic of Conditionals", (forthcoming).

RAMSEY, F. P. ([1929] 1990): "General Propositions and Causality", in Mellor, D. H. (ed.) *Frank Plumpton Ramsey: Philosophical Papers*, Cambridge: Cambridge University Press, pp. 145–63.

SCHAY G. (1968): "An Algebra of Conditional Events", *Journal of Mathematical Analysis and Applications*, 24, pp. 334–44.