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Gödel Łukasiewicz Logic

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ABSTRACT. This paper studies many-valued logic endowed with two different kinds of implications: Łukasiewicz implication and Gödel implication. This research focuses on the class of algebras containing the algebraic counterpart of this new logic: the class of Heyting Wajsberg algebras. We prove that this variety is a discriminator variety. We show Gödel Łukasiewicz Logic to be regularly algebraizable, strongly complete, decidable and to have the Deduction-Detachment Theorem.

KEYWORDS: many valued logics, algebraizable logics, bounded distributive lattices, MV-algebras, HW-algebras, discriminator variety, equationally definable principal congruences, strong completeness, subdirectly irreducible algebras.

Introduction

Many valued logic arose by the study of Jan Łukasiewicz who first, in the early twenties of the last century, understood the importance of employing an infinite set of truth values in the semantics of a deductive system. Once interpreted this set of values as the unit interval of real numbers, he proposed the following interpretation for the implicational connective:

$$x \to_L y := \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{otherwise} \end{cases}$$

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Later C. C. Chang [14] in order to prove the completeness of the axioms of Łukasiewicz $\aleph_0$-valued propositional calculus proved the Lindenbaum-Tarski algebra of this deductive system to be an MV-algebra. Many other interpretations of the implicational connective have been introduced in the scientific literature.

After Lotfi-A. Zadeh conceived fuzzy set theory [25] a renewed interest pushed philosophers and mathematicians to investigate the possible ways to enlarge the previous studies in many valued logic. A long debate on which connective is the most appropriate to interpreter intersection in fuzzy sets took place during several years and then the scientific community seemed to agree to define the intersection operator (and its dual union) in an axiomatic way as a class: the set of triangular norms (and its dual conorms) [26]. A triangular norm is any continuous function $t : [0,1]^2 \mapsto [0,1]$ that satisfies the following four properties: it has to be commutative, associative, monotonic and to have 1 as neutral element. Moreover by the residuation law $x \wedge y \leq z \Leftrightarrow x \leq y \to z$ any triangular norm is associated to an implication (i.e. its residuum). The class of the triangular norms is a range of functions limited at the top by the minimum (i.e. the maximal triangular norm) and at the bottom by the Łukasiewicz triangular norm t_L (i.e. the minimal triangular norm) defined for any $x,y \in [0,1]$, $xt_Ly := \min\{0, x+y-1\}$. The residuation law connects Łukasiewicz triangular norm with Łukasiewicz implication and minimum with the following implicational connective:

$$x \to_G y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

This implication has been introduced by Kurt Gödel in [18] and then is usually known as Gödel implication. Hence, Gödel implication and Łukasiewicz implication are the two residua delimiting the whole range of triangular norms. For this reason we want to study many-valued logic endowed with these two different implications as primitive operators.

1. The logical system

Let $\Lambda := < \to_G, \to_L, \mathbf{0} >$ be a language of type $< 2,2,0 >$. Gödel Łukasiewicz Logic, in symbols $GLL := < \Lambda, \vdash_{GLL} >$ is the deductive system presented by the following collection of axioms (Ax1-8) and inference rule (MP1). First we recall, for the sake of readability, the non-primitive connectives definitions:

$$\begin{array}{rcl} \neg\alpha & := & \alpha \to_L \mathbf{0} \\ \sim\alpha & := & \alpha \to_G \mathbf{0} \\ \alpha \wedge \beta & := & \neg((\neg\alpha \to_L \neg\beta) \to_L \neg\beta) \\ \alpha \vee \beta & := & (\beta \to_L \alpha) \to_L \alpha = \neg(\neg\alpha \wedge \neg\beta) \\ \alpha \leftrightarrow \beta & := & (\alpha \to_L \beta) \wedge (\beta \to_L \alpha) \end{array}$$

(Ax1) $\alpha \to_G \alpha$
(Ax2) $(\alpha \to_G (\beta \wedge \gamma)) \leftrightarrow ((\alpha \to_G \gamma) \wedge (\beta \to_G \gamma))$
(Ax3) $(\alpha \wedge (\alpha \to_G \beta)) \leftrightarrow (\alpha \wedge \beta)$
(Ax4) $((\alpha \vee \beta) \to_G \gamma) \leftrightarrow ((\alpha \to_G \gamma) \wedge (\beta \to_G \gamma))$
(Ax5) $((\alpha \to_G \alpha) \to_L \alpha) \leftrightarrow \alpha$
(Ax6) $(\alpha \to_L (\beta \to_L \gamma)) \leftrightarrow (\neg(\alpha \to_L \gamma) \to_L \neg\beta)$
(Ax7) $\neg \sim \alpha \to_L \sim\sim \alpha$
(Ax8) $(\alpha \to_G \beta) \to_L (\alpha \to_L \beta)$
(MP1) $\frac{\alpha, \alpha \to_L \beta}{\beta}$

Definition 1.1. MP2 $:= \frac{\alpha, \alpha \to_G \beta}{\beta}$.

Lemma 1.1. MP1 $\Rightarrow$ MP2.

Proof. By (Ax8) and two applications of MP1. □

Definition 1.2. Let us consider the set of well formed formulas of *GLL* defined by induction in the traditional way, in symbols $Fm(\Lambda)$. An evaluation on $Fm(\Lambda)$ is a mapping $e : Fm(\Lambda) \mapsto [0,1]$ such that:

$$e(\alpha \to_L \beta) = \min\{1, 1 - e(\alpha) + e(\beta)\}$$

$$e(\alpha \to_G \beta) = \begin{cases} 1 & \text{if } e(\alpha) \leq e(\beta) \\ e(\beta) & \text{otherwise} \end{cases}$$

Definition 1.3. A formula $\upsilon \in Fm(\Lambda)$ is a 1-tautology iff $e(\upsilon) = 1$ for any evaluation e.

2. Basic algebraic notions

Heyting algebras are the algebraic conterpart (i.e. the class of algebraic structures which verify exactly the provable formulae) of intuitionistic propositional logic (pp. 380-383 [15]). In the same way Wajsberg algebras are the algebraic counterpart of the $\aleph_0$-valued Łukasiewicz propositional calculus and Wajsberg algebras are termwise definitionally equivalent to MV-algebras [13].

The class of Heyting Wajsberg algebras has been first introduced by Giampiero Cattaneo and Davide Ciucci [9]. They explained that by the composition of the two primitive operators with the **0**-element it is possible to define the modal-like operators of necessity and possibility (and their duals). Moreover the pairs of these unary operators generate rough approximation spaces of Boolean elements and Heyting Wajsberg algebras define an abstract enviroment linking fuzzy and rough sets.

In the sequel we are going to show that the set of 1-tautologies of Gödel Łukasiewicz logic (i.e. the logic whose the algebraic equivalent semantics is the class of Heyting Wajsberg algebras) contains both the one of intuitionistic propositional logic and the one of Łukasiewicz many valued propositional logic. In order to do that we will prove that the same relationships among the corresponding equational theories hold. First we have to report some well known definitions.

Definition 2.1. A Heyting algebra is a structure $\mathscr{A} = \langle A, \wedge, \vee, \rightarrow, 0\rangle$ of type $\langle 2,2,2,0\rangle$ in which the following axioms are satisfied:

(H1) $x \rightarrow x = y \rightarrow y$
(H2) $(x \rightarrow y) \wedge y = y$
(H3) $x \rightarrow (y \wedge z) = (x \rightarrow z) \wedge (x \rightarrow y)$
(H4) $x \wedge (x \rightarrow y) = x \wedge y$
(H5) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
(H6) $0 \wedge x = 0$

In [23] it is proved that Heyting algebras are equivalent to residuated lattices where the multiplication operator coincides with the lattice meet operator, i.e., $\forall x, y \in A : x \wedge y = x * y$. Moreover the unary operator $\sim x := x \rightarrow 0$ is a Brouwer negation, i.e., it satisfies the three following properties:

(B1) $x \leq \sim\sim x$
(B2) $\sim (x \vee y) = \sim x \wedge \sim y$

(B3) $x \wedge \sim x = 0$

Definition 2.2. A Wajsberg algebra is a structure $\mathscr{A} = \langle A, \rightarrow, \neg, 1\rangle$ of type $\langle 2,1,0\rangle$ in which the following axioms are satisfied:

(W1) $1 \rightarrow x = x$
(W2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$
(W3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
(W4) $(\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x) = 1$

Let us remark that in any Wajsberg algebra it is possible to define an order relation as

$$a \leq b \quad \text{iff} \quad a \rightarrow b = 1$$

With respect to this order relation, 1 is the maximum element and $0 := \neg 1$ the minimum element and once defined the meet and join operators in the usual way: $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$, the structure $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice.

Definition 2.3. An *MV-algebra* is a structure $\mathscr{A} = \langle A, \oplus, \neg, \mathbf{0}\rangle$ of type $\langle 2,1,0\rangle$ such that, upon defining for any $x, y \in A : x \vee y := \neg(\neg x \oplus y) \oplus y$, the following conditions are required:

(MV1) $(x \oplus y) \oplus z = (y \oplus z) \oplus x$
(MV2) $x \oplus \mathbf{0} = x$
(MV3) $x \oplus \neg\mathbf{0} = \neg\mathbf{0}$
(MV4) $x \vee y = y \vee x$
(MV5) $\neg\neg x = x$

It is useful to define also the dual concepts: $\mathbf{1} := \neg\mathbf{0}$, $x \odot y := \neg(\neg x \oplus \neg y)$, and $x \wedge y := \neg(\neg x \odot y) \odot y$. We observe that the relation $x \leq y \Leftrightarrow x \vee y = y$ induces in every MV-algebra a distributive lattice order. In what follows we denote by $\mathscr{A}_{[0,1]}$ the standard MV-algebra whose support is the unit real interval and by $\mathscr{A}_{[0,1]\cap Q}$ the algebra whose support is the unit rational interval. Notice that in both these algebras for any x, y in their support, $\neg x := 1 - x$, $x \oplus y := \min\{1, x+y\}$ and $\mathbf{0} := 0$.

Definition 2.4. An MV-algebra $\mathscr{A}$ is a *Stonean MV-algebra* if and only if for any $x \in A$, there exists a Boolean (i.e. idempotent) element z such that $z = \bigvee\{y \mid y \wedge x = \mathbf{0}\}$. This property is equivalent to the possibility to define a Stonean negation $\sim$, for any $x \in A, \sim x := z$. Moreover a Stonean MV-algebra is an algebra $\mathscr{A} = \langle A, \oplus, \neg, \sim, \mathbf{0}\rangle$ of type $\langle 2,1,1,0\rangle$.

Definition 2.5. Let $\mathscr{A} = \langle A, \to_L, \to_G, \mathbf{0}\rangle$ be an algebraic structure of type $\langle 2,2,0\rangle$. $\mathscr{A}$ is a *Heyting Wajsberg algebra* if for any $x,y,z \in A$, once defined

$$\begin{array}{rcl} \neg x & := & x \to_L \mathbf{0} \\ \sim x & := & x \to_G \mathbf{0} \\ x \wedge y & := & \neg((\neg x \to_L \neg y) \to_L \neg y) \\ x \vee y & := & (x \to_L y) \to_L y \\ \mathbf{1} & := & \neg \mathbf{0} \end{array}$$

the following identities are satisfied:

(HW1) $x \to_G x = \mathbf{1}$
(HW2) $x \to_G (y \wedge z) = (x \to_G z) \wedge (x \to_G y)$
(HW3) $x \wedge (x \to_G y) = x \wedge y$
(HW4) $(x \vee y) \to_G z = (x \to_G z) \wedge (y \to_G z)$
(HW5) $\mathbf{1} \to_L x = x$
(HW6) $x \to_L (y \to_L z) = \neg(x \to_L z) \to_L \neg y$
(HW7) $\neg \sim x \to_L \sim\sim x = \mathbf{1}$
(HW8) $(x \to_G y) \to_L (x \to_L y) = \mathbf{1}$

Let us introduce the following conventions:

$$\begin{array}{rcl} x \oplus y & := & \neg x \to_L y \\ x \odot y & := & \neg(\neg x \oplus \neg y) \end{array}$$

Any HW-algebra $\mathscr{A} = \langle A, \to_L, \to_G, \mathbf{0}\rangle$ has the MV-algebra $\mathscr{A}^\star = \langle A, \oplus, \neg, \mathbf{0}\rangle$ as term reduct and any HW-algebra $\mathscr{A} = \langle A, \to_L, \to_G, \mathbf{0}\rangle$ has the bounded distributive lattice $\mathscr{A}^{\star\star} = \langle A, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ as term reduct ([10], [11]).

It is also shown in [11] (proposition 1.1) that the natural partial order $\leq$ defined by $\wedge$ or $\vee$ (*i.e.* $x \leq y := x \wedge y = x$ or $x \leq y := x \vee y = y$) has the following property:

(P) $x \leq y \Leftrightarrow x \to_L y = \mathbf{1} \Leftrightarrow x \to_G y = \mathbf{1}$

Proposition 2.1. In any linear MV-algebra a Stonean negation can be defined in a natural way in order to have a Heyting Wajsberg algebra term reduct.

Proof. By [11] any HW-algebra is termwise definitionally equivalent to a Stonean MV-algebra. An MV-algebra is Stonean when there can be defined a Stonean negation (see also [12]). Any linear MV-algebra is trivially Stonean once defined the Stonean negation $\sim_0$:

$$\sim_0 x := \begin{cases} \mathbf{0} & \text{if } x \neq \mathbf{0} \\ \mathbf{1} & \text{if } x = \mathbf{0} \end{cases}$$

Then any linear MV-algebra enriched in such a way has a HW-algebra term reduct. □

A HW-algebra $\mathscr{A}$ is *linear (or totally ordered)* iff for any pair of elements $x, y \in A$, either $x \leq y$ or $y \leq x$. A linear algebra can be also called for short a *chain*.

Now we introduce the most important example of Heyting Wajsberg algebra, the model we will prove to generate the whole set of Heyting Wajsberg algebras as quasi-variety.

Example 2.1 (Standard HW-algebra)**.** The algebra whose support represents the set of truth-values of *GLL* is

$$\mathscr{A}_{[0,1]} = \langle [0,1], \rightarrow_L, \rightarrow_G, 0 \rangle$$

where $[0,1] \subset R$, and the two binary operators are defined

$$a \rightarrow_L b := \min\{1, 1 - a + b\}$$

$$x \rightarrow_G y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Another important example of Heyting Wajsberg algebra in order to study *GLL* follows.

Example 2.2 (Lindenbaum-Tarski algebra of *GLL*)**.** Let the binary relation $\equiv$ on $Fm(\Lambda)$ be defined by $\alpha \equiv \beta$ iff $\vdash_{GLL} \alpha \rightarrow_L \beta$ and $\vdash_{GLL} \beta \rightarrow_L \alpha$. Then $\equiv$ is a congruence relation and the quotient set $Fm(\Lambda)_{\equiv}$ becomes a HW-algebra with the operation $\rightarrow_L$, $\rightarrow_G$ and the constant $\perp$ defined by

$$[\alpha]_{\equiv} \rightarrow_L [\beta]_{\equiv} := [\alpha \rightarrow_L \beta]_{\equiv}$$

$$[\alpha]_{\equiv} \rightarrow_G [\beta]_{\equiv} := [\alpha \rightarrow_G \beta]_{\equiv}$$

$$\perp := \{\gamma \mid \vdash_{GLL} \beta \text{ and } (\beta \rightarrow_L \mathbf{0}) \equiv \gamma\}$$

The reader can find a self contained proof of the standard completeness of Heyting Wajsberg algebras in [21]. However this result could be also inferred in an indirect way by the connection between Heyting Wajsberg algebras and Stonean MV-algebras [11] because in [20] it is proved that the variety of Stonean MV-algebras is generated by the model whose support is the unitary real interval (i.e. the standard Stonean MV-algebra). We report this important result.

Theorem 2.3. $\mathbf{HW} = HSP(\mathscr{A}_{[0,1]})$.

In the sequel we will adopt the following notation. Given a HW-algebra $\mathscr{A}$, $\forall x \in A$ and $\forall n \in N$:

$$nx = \begin{cases} \mathbf{0} & \text{if } n = 0 \\ x & \text{if } n = 1 \\ \underbrace{x \oplus \ldots \oplus x}_{n-times}, & \text{if } 2 \leq n \in N \end{cases}$$

and

$$x^n = \begin{cases} \mathbf{1} & \text{if } n = 0 \\ x & \text{if } n = 1 \\ \underbrace{x \odot \ldots \odot x}_{n-times}, & \text{if } 2 \leq n \in N \end{cases}$$

We recall below an important basic result that will be useful in the sequel.

Lemma 2.1. Let $\mathscr{A}$=$\langle A, \to, \neg, \mathbf{0} \rangle$ be a Wajsberg algebra. Then for any $x, y, x \in A$ the following properties hold:

1. $x \to (y \to z) = y \to (x \to z)$
2. $x \to y = \neg y \to \neg x$

Proof. See Lemma 4.2.3 and Lemma 4.2.4 in [13]. □

In the next lemma and corollary Heyting, Wajberg and Heyting Wajsberg algebras are dealt in signatures that are not the shortest (i.e. with the smallest number of symbols) in which they had been defined previously but these signatures are however their extensions. This can be found often in the algebraic literature and we decide to adopt it for the sake of a clear wide comprehension. We advise the reader to consult also Lemma 16 in [16] because of its strict connection with the following lemma.

Lemma 2.2. An algebra $\mathscr{A} = \langle A, \wedge, \vee, \rightarrow_L, \neg, \rightarrow_G, \sim, \mathbf{0}, \mathbf{1} \rangle$ of type $\langle 2,2,2,1,2,1,0,0 \rangle$ has a Heyting Wajsberg term reduct if and only if the following conditions are satisfied:

1. $\langle A, \rightarrow_L, \neg, \mathbf{0}, \mathbf{1} \rangle$ is a Wajsberg algebra.
2. $\langle A, \wedge, \vee, \rightarrow_G, \sim, \mathbf{0}, \mathbf{1} \rangle$ is a Heyting algebra.
3. The Wajsberg algebra partial order $\leq^{\rightarrow_L}$ $(a \leq^{\rightarrow_L} b := a \rightarrow_L b = \mathbf{1})$ and the Heyting algebra partial order $\leq^{\rightarrow_G}$ $(a \leq^{\rightarrow_G} b := a \rightarrow_G b = \mathbf{1})$ coincide.

Proof. We prove separately the two implications.

($\Rightarrow$) For the left-to-right direction, let $\mathscr{A}$ be a HW-algebra.

(1) In [11] it is shown that Heyting Wajsberg algebras are termwise definitionally equivalent to Stonean MV-algebras that are trivially MV-algebras. MV-algebras are proved to be termwise definitionally equivalent to Wajsberg algebras [13]. Thus any Heyting Wajsberg algebra satisfies the axioms of Wajsberg algebras.

(2) We have to prove that the Heyting Wajsberg algebra $\mathscr{A}$ satisfies the axioms (H1)-(H6). (H1) follows easily from (HW1). (H3), (H4) and (H5) coincide with (HW2), (HW3) and (HW4). Since by Theorem 2.3 we have that **HW** = HSP($\mathscr{A}_{[0,1]}$), (H2) can be just verified in $\mathscr{A}_{[0,1]}$ where either $x \leq y$ or $y < x$ and where we remind that $\rightarrow_G$ is defined

$$x \rightarrow_G y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Since any Heyting Wajsberg algebra $\mathscr{A}$ has a bounded distributive lattice $\mathscr{A}^{\star\star}$ term reduct (H6) is satisfied.

(3) This follows from (P).

($\Leftarrow$) For the right-to-left direction, suppose $\mathscr{A}$ satisfies (1)-(3). By (W2), $(\mathbf{1} \rightarrow_L \mathbf{1}) \rightarrow_L ((\mathbf{1} \rightarrow_L x) \rightarrow (\mathbf{1} \rightarrow_L x)) = \mathbf{1}$ and then by (W1), $x \rightarrow_L x = \mathbf{1} \rightarrow_L (x \rightarrow_L x) = \mathbf{1}$. By (P) we have (HW1). It can be easily observed that (HW2), (HW3), (HW4), (HW5) correspond to (H3), (H4), (H5) and (W1). By combining the two statements in Lemma 2.1 (HW6) holds in any Wajsberg algebra.

One of the basic property in an MV-algebra is that for all $x, y \in A$ $x \odot y \leq x \wedge y$. By (H4)/(HW3) $y \geq x \wedge y = x \wedge (x \rightarrow_G y) \geq x \odot (x \rightarrow_G y)$. The Łukasiewicz implication $\rightarrow_L$ is a residuum with respect to $\odot$, then by residuation law we have $(x \rightarrow_G y) \odot x \leq y \Leftrightarrow (x \rightarrow_G y) \rightarrow_L (x \rightarrow_L y) = \mathbf{1}$ that is (HW8).

We need a remark in order to prove (HW7). Any Wajsberg algebra is termwise definitionally equivalent to an MV-algebra that is isomorphic to a subdirect product of a family of linear MV-algebras $\{\mathscr{A}_i \mid i \in I\}$ [14]. Suppose $\sim x = y$ and that in the subdirect representation each element $z \in A$, is expressed componentwise $z = \langle z_1, \ldots, z_n, .. \rangle$ where $1 \leq n \leq i \in I$. We distinguish two cases:

1. By $x \wedge y = \mathbf{0}$ (B3) if $x_i \neq \mathbf{0}_i$ then $y_i = \mathbf{0}_i$.
2. Let $x_i = \mathbf{0}_i$.
   2.1. Suppose $y_i \neq \mathbf{0}_i$. Let us consider the set of element $\{z^j \mid j \in J\}$ such that for any $j \in J$, $x \wedge z^j = \mathbf{0}$ and $z_i^j \neq \mathbf{0}_i$. This set is not empty because y belongs to it.
   By (B3) for any $j \in J$, $\sim z_i^j = b_i = \mathbf{0}_i = x_i$.
   By (B1) for any $j \in J$, $z_i^j \leq \sim\sim z_i^j = \sim b_i = \sim x_i = y_i$ and hence y_i has to be the maximal element of $\{z_i^j \mid j \in J\}$.
   Moreover y_i has to be $\oplus$-idempotent because otherwise $k = 2y > y$ and $y_i < k_i \in \{z_i^j \mid j \in J\}$ against maximality. Since a linear MV-algebra has the only two idempotent elements $\mathbf{0}$ and $\mathbf{1}$, $y_i = \mathbf{1}_i$.
   2.2. $y_i = \mathbf{0}_i$.

This means that in any case y is an idempotent element, i.e. $\sim x \oplus \sim x = \sim x$. By a well known MV-property (Theorem 1.5.3 in [13]) if $y \oplus y = y$ then $\neg y \wedge y = \mathbf{0}$ and then we obtain $\sim x \wedge \neg \sim x \leq \mathbf{0}$ that by residuation law is equivalent to $\neg \sim x \leq \sim\sim x$. By hypothesis 3. we have (HW7).

□

Corollary 2.1. Heyting-Wajsberg algebras are axiomatized in the signature $\langle \wedge, \vee, \rightarrow_L, \neg, \rightarrow_G, \sim, \mathbf{0}, \mathbf{1} \rangle$ by:

1. A set of identities axiomatizing Heyting algebras in the signature $\langle \wedge, \vee, \rightarrow_G, \sim, \mathbf{0}, \mathbf{1} \rangle$.
2. A set of identities axiomatizing Wajsberg algebras in the signature $\langle \rightarrow_L, \neg, \mathbf{1} \rangle$.
3. The identity $x \vee y = (x \rightarrow_L y) \rightarrow_L y$.

From the previous lemma we conclude that Heyting Wajsberg algebras satisfy the following useful identities, which will be needed in the sequel:

$$x \rightarrow_L x = \mathbf{1}$$

$$x \rightarrow_L \mathbf{1} = \mathbf{1}$$

3. Algebraic properties

We recall the fundamental notion of discriminator variety on which a wide literature exists (see for example [4]).

Definition 3.1. A *discriminator term* on a set A is a function $t : A^3 \mapsto A$ such that, for any $a, b, c \in A$:

$$t(a,b,c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases}$$

Definition 3.2. An algebra $\mathscr{A}$ is *subdirectly irreducible* if for every subdirect embedding $f : \mathscr{A} \mapsto \prod_{i \in I} \mathscr{A}_i$ there is an $i \in I$ such that $\pi_i \circ f : \mathscr{A} \mapsto \mathscr{A}_i$ is an isomorphism.

Definition 3.3. A variety $\mathbf{V}$ is said to be a *discriminator variety* if there exists a ternary term t such that t is a discriminator term on each subdirectly irreducible member of $\mathbf{V}$. A *pointed discriminator variety* is a discriminator variety with a constant term.

Now we can state a theorem that opens the path to many important other results:

Theorem 3.1. **HW** is a discriminator variety.

Proof. Since by Theorem 2.3 $\mathbf{HW} = HSP(\mathscr{A}_{[0,1]})$ in order to prove **HW** to be a discriminator variety we have just to find a ternary term that is a discriminator term on the standard HW-algebra $\mathscr{A}_{[0,1]}$. First we define:

$$\tau(a,b) := \neg \sim \neg((a \rightarrow_G b) \wedge (b \rightarrow_G a))$$

It can be easily verified that

$$\tau(a,b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{otherwise} \end{cases}$$

Let us consider the following ternary term

$$\sigma(a,b,c) := (\tau(a,b) \wedge a) \vee (\sim \tau(a,b) \wedge c)$$

If $a = b$, $\tau(a,b) = 0$ and $\sigma(a,b,c) = (0 \wedge a) \vee (1 \wedge c) = c$. If $a \neq b$, $\tau(a,b) = 1$ and $\sigma(a,b,c) = (1 \wedge a) \vee (0 \wedge c) = a$. We have proved σ to be a discriminator term on $\mathscr{A}_{[0,1]}$, then on any member of **HW** and hence **HW** is a discriminator variety. □

A different discriminator term for HW-algebras has been provided in [2] but it has never been published. The property to be a discriminator variety opens the path to many related algebraic consequences. When such a variety has equationally definable principal congruences, if it is a variety generated by the real unit interval model so it is as quasi-variety: quasi-equations satisfied in the prototypical model are satisfied in any member of the variety. This result is known as strong completeness theorem.

In MV-algebras there is a one-one correspondence between ideals and congruences. To have this relationship in HW-algebras, in [21] a new adequate definition of filter (the dual concept of ideal) has been introduced. We are going to recall some basic definitions to introduce the concept of equationally definable principal congruences (i.e.EDPC) in order to prove the strong completeness theorem for HW-algebras with respect to the standard unit interval HW-algebra. On EDPC the four works of W.Blok and D.Pigozzi quoted in the references can give to the reader a deep and exhaustive insight.

Definition 3.4. A variety **V** is congruence distributive (modular) if and only if for any $\mathscr{A} \in \mathbf{V}$, the lattice of congruences (i.e. Con(A)) is a distributive (modular) lattice.

Definition 3.5. An algebra $\mathscr{A}$ has the *congruence extension property* (CEP) if for every subalgebra of the same class $\mathscr{B}$ and any congruence $\theta \in \mathrm{Con}(B)$ there is a congruence $\phi \in \mathrm{Con}(A)$ such that $\theta = \phi \cap B^2$ where B^2 is the set of all the 2-tuples of elements from B. A class K of algebras has the CEP if every algebra in the class has the CEP.

Definition 3.6. Let $\mathscr{A}$ be an algebra and $a_1,\ldots,a_n \in A$, let $\Theta(a_1,\ldots,a_n)$ be a congruence generated by $\{\langle a_i,a_j\rangle \mid 1 \leq i,j \leq n\}$. i.e. the smallest congruence such that $a_1,\ldots,a_n$ are in the same class. The congruence $\Theta(a_1,a_2)$ is called *principal congruence.*

Definition 3.7. A class of algebras **K** is said to have *equationally definable principal congruences* (briefly EDPC) if there exists a finite set of quaternary terms $p_i(x,y,z,w)$, $q_i(x,y,z,w)$ of **K** such that for every algebra $\mathscr{A} \in \mathbf{K}$ and all $a,b,c,d \in A$,

$$c \equiv d(\mathrm{mod}\Theta^A(a,b)) \text{ if and only if}$$
$$p_i^A(a,b,c,d) = q_i^A(a,b,c,d), \text{ for } i = 1,\ldots,n.$$

Block and Pigozzi (Theorem 3.8 in [5]) proved every discriminator variety to have EDPC. We give an explicit proof for Heyting Wajsberg algebras for the sake of readability.

Theorem 3.2. The variety of Heyting Wajsberg algebras has EDPC.

Proof. In the previous theorem we have proved **HW** to be a discriminator variety. Then for any $\mathscr{A} \in$ **HW** there is a discriminator ternary term, such that for any $a,b,c \in A$:

$$t(a,b,c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases}$$

We can define a couple of quaternary terms p_1 and q_1:

$$p_1(a,b,c,d) := (d \to_G d) \to_G t(a,b,c)$$

$$q_1(a,b,c,d) := (c \to_G c) \to_G t(a,b,d)$$

Then **HW** has EDPC. □

Corollary 3.1. HW is congruence distributive (and thus congruence modular) and has the congruence extension property.

Proof. See Theorem 1.2 in [8]. □

The following theorem is part of the folklore of universal algebra. A clear presentation of this result can also be found in [1].

Theorem 3.3. Let K be a class of simple algebras such that the variety generated by K (we denote it $V(K)$) has EDPC. $V(K)$ coincides with the quasi-variety generated by K. In symbols, HSP(K) = ISPPu(K).

Theorem 3.4 (Strong completeness)**. HW** = ISPPu$(\mathscr{A}_{[0,1]})$.

Proof. By Theorem 2.3 **HW**= HSP($\mathscr{A}_{[0,1]}$) and in Theorem 3.2 it is shown **HW** to have EDPC. $\mathscr{A}_{[0,1]}$ is trivially simple, then by Theorem 3.3 the standard HW-algebra generates the quasi-variety **HW**. □

Now we are going to show the finite model property for Heyting Wajsberg algebras. The following lemma and theorems provide the necessary steps.

It is important to remind that, by the classical construction, any real number can be identified by a sequence of Cauchy of rational numbers. I recall this definition:

Definition 3.8. A sequence $a : n \mapsto a_n$ (we write $\{a_n\}$) is a Sequence of Cauchy if and only if $\quad \forall \varepsilon > 0, \quad \exists \nu, \quad \forall m,n > \nu, \quad d(a_m, a_n) < \varepsilon.$

I am going to show that sequences of Cauchy are closed under the sum.

Lemma 3.1. Let $a : n \mapsto a_n$ and $b : n \mapsto b_n$ two sequences of Cauchy. Then $a + b : n \mapsto a_n + b_n$ is a sequence of Cauchy.

Proof. By definition we have directly that $\forall \varepsilon > 0, \exists \nu_1, \forall m,n > \nu_1, d(a_m, a_n) < \frac{\varepsilon}{2}$ and $\forall \varepsilon > 0, \exists \nu_2, \forall m,n > \nu_2, d(b_m, b_n) < \frac{\varepsilon}{2}$. Then $\forall \varepsilon > 0, \exists \nu_i =max\{\nu_1, \nu_2\}, \forall m,$ $n > \nu_i, d(a_m + b_m, a_n + b_n) \leq d(a_m, a_n) + d(b_m, b_n) < \varepsilon.$ □

It can be easily observed that whether a sum is truncated (for instance, $\oplus$ in $\mathscr{A}_{[0,1]\cap Q}$) the property expressed by the previous lemma is not affected and it still holds. Moreover $\{0_n\}$ and $\{1_n\}$ are trivially sequences of Cauchy. Then $\neg(n : n \mapsto a_n)$ is $\{1_n\}$ + $((-(n : n \mapsto a_n)) := (n : n \mapsto -a_n))$ and sequences of Cauchy are closed under $\oplus$ and $\neg$ componentwise.

The following theorem is part of the MV folklore but an explicit proof cannot be found in the literature. Then we have decided to prove and present it.

Theorem 3.5. Let $\mathscr{A}$ be an MV-algebra, HSP$(\mathscr{A}_{[0,1]\cap Q}) =$ HSP$(\mathscr{A}_{[0,1]})$.

Proof. Since $\mathscr{A}_{[0,1]\cap Q}$ is an MV-algebra and **MV** = HSP$(\mathscr{A}_{[0,1]})$ [13], we have $\mathscr{A}_{[0,1]\cap Q} \in$ HSP$(\mathscr{A}_{[0,1]})$ and then HSP$(\mathscr{A}_{[0,1]\cap Q}) \subseteq$ HSP$(\mathscr{A}_{[0,1]})$. On the other hand HSP$(\mathscr{A}_{[0,1]\cap Q})$ is closed under direct products, then an MV-algebra $\mathscr{A}^{\omega}_{[0,1]\cap Q}$ whose support is made of infinite copies of $\mathscr{A}_{[0,1]\cap Q}$ and whose operators are defined componentwise, belongs to HSP $(\mathscr{A}_{[0,1]\cap Q})$. This variety is also closed under subalgebras. The set of the sequences of Cauchy in $[0,1] \cap Q$ endowed with componentwise-defined truncated sum and involutive negation $\mathscr{A}^{\{a_n\}}_{[0,1]\cap Q}$ is an MV-subalgebra of $\mathscr{A}^{\omega}_{[0,1]\cap Q}$. Then $\mathscr{A}^{\{a_n\}}_{[0,1]\cap Q} \in$ HSP$(\mathscr{A}_{[0,1]\cap Q})$. Since real numbers are identified by sequences of Cauchy there is trivially a homomorphism from $\mathscr{A}^{\{a_n\}}_{[0,1]\cap Q}$ to $\mathscr{A}_{[0,1]}$. $\mathscr{A}_{[0,1]}$ is the quotient of $\mathscr{A}^{\{a_n\}}_{[0,1]\cap Q}$ modulo the congruence R defined by $aRb :=$ a and b have the same limit. Thus $\mathscr{A}_{[0,1]} \in$ HSP$(\mathscr{A}_{[0,1]\cap Q})$ and HSP$(\mathscr{A}_{[0,1]}) \subseteq$ HSP$(\mathscr{A}_{[0,1]\cap Q})$. Then HSP$(\mathscr{A}_{[0,1]\cap Q}) =$ HSP$(\mathscr{A}_{[0,1]})$. □

Theorem 3.6. Let $\mathscr{A}$ be a Stonean MV-algebra, HSP$(\mathscr{A}_{[0,1]\cap Q}) =$ HSP$(\mathscr{A}_{[0,1]})$.

Proof. Since in [20] it is proved the variety of Stonean MV-algebras **SMV** = HSP($\mathscr{A}_{[0,1]}$) we have HSP($\mathscr{A}_{[0,1]\cap Q}$) $\subseteq$ HSP($\mathscr{A}_{[0,1]}$). We have to prove that HSP($\mathscr{A}_{[0,1]}$) $\subseteq$ HSP($\mathscr{A}_{[0,1]\cap Q}$) that is, if $\mathscr{A}_{[0,1]\cap Q} \models t(x_1,\ldots,x_n) = s(x_1,\ldots,x_m)$ then $\mathscr{A}_{[0,1]} \models t(x_1,\ldots,x_n) = s(x_1,\ldots,x_m)$. We'll do it by induction on the number of occurences of $\sim$ in $t \cup s$ (we denote it $\sim$-occ($t \cup s$)). If $\sim$-occ($t \cup s$)= 0 then $t(x_1,\ldots,x_n) = s(x_1,\ldots,x_m)$ is an MV-equation and by Theorem 3.5, if $\mathscr{A}_{[0,1]\cap Q} \models t(x_1,\ldots,x_n) = s(x_1,\ldots,x_m)$ then $\mathscr{A}_{[0,1]} \models t(x_1,\ldots,x_n) = s(x_1,\ldots,x_m)$. By hypothesis of induction we suppose that this implication holds for each $j \leq k$ = $\sim$-occ($t \cup s$). If $\sim$-occ($t \cup s$)= $k+1$, for some $1 \leq i \leq n$, without loss of generality we have that $\mathscr{A}_{[0,1]\cap Q} \models t(x_1,\ldots,x_n,\sim t'(x_1,\ldots,x_n)) = s(x_1,\ldots,x_m)$ and $\sim$-occ($t' \cup s$) $\leq k$. Since $\sim$ in both $\mathscr{A}_{[0,1]\cap Q}$ and $\mathscr{A}_{[0,1]}$ is $\sim_0$ and

$$\sim_0 (a) = \begin{cases} 0 & \text{if } a \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

there will be a term $t^\star$ for which, by induction hypothesis, if $\mathscr{A}_{[0,1]\cap Q} \models t^\star(x_1,\ldots, x_n, 0) = s(x_1,\ldots,x_m)$ then $\mathscr{A}_{[0,1]} \models t^\star(x_1,\ldots,x_n,0) = s(x_1,\ldots,x_m)$ and if $\mathscr{A}_{[0,1]\cap Q} \models t^\star(x_1,\ldots,x_n,1) = s(x_1,\ldots,x_m)$ then $\mathscr{A}_{[0,1]} \models t^\star(x_1,\ldots,x_n,1) = s(x_1,\ldots,x_m)$. It follows $\mathscr{A}_{[0,1]} \models t(x_1,\ldots,x_n,\sim t'(x_1,\ldots,x_n)) = s(x_1,\ldots,x_m)$. □

An important feature in the interconnection between logic and algebra is the following: any logical finitely axiomatized propositional calculus is decidable if its Lindenbaum-Tarski algebra belongs to a variety that has the *finite model property* (FMP). We introduce this definition.

Definition 3.9. We say that a variety has the finite model property (FMP) if every identity that fails to hold in the class can be refuted in a finite member of the class. Varieties with FMP are said to be *generated by their finite members*.

Throughout this section, for each $n = 1,2,3,...$ we shall use the notation:

$$\mathbf{Z}\frac{1}{n \cap [0,1]} := \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

Moreover we denote with $\mathscr{A}_{n+1}$ the subalgebra of $\mathscr{A}_{[0,1]\cap Q}$ whose support is $\mathbf{Z}\frac{1}{n\cap[0,1]}$.

Theorem 3.7. The variety of Stonean MV-algebras (**SMV**) has FMP.

Proof. By Theorem 3.6 and since **SMV** = HSP($\mathscr{A}_{[0,1]}$) [20], if an equation α : $t = s$ in the variables $x_1,\dots,x_n$ fails in some Stonean MV-algebra $\mathscr{A}$ then it fails in $\mathscr{A}_{[0,1]\cap Q}$. Hence for some $c_1,\dots,c_n \in [0,1]\cap Q$, $t^{A_{[0,1]\cap Q}}(c_1,\dots,c_n) \neq s^{A_{[0,1]\cap Q}}(c_1,\dots,c_n)$. So if $c_1 \in \mathscr{A}_{m+1}, c_2 \in \mathscr{A}_{n+1},\dots,c_n \in \mathscr{A}_{v+1}$ and $\mathscr{A}_{ij}$ is the subalgebra of $\mathscr{A}_{[0,1]\cap Q}$ whose support is $\mathbf{Z}_{\frac{1}{m\cdot n\cdot\ldots\cdot v}\cap[0,1]}$ then $t^{A_{ij}}(c_1,\dots,c_n) \neq s^{A_{ij}}$ $(c_1,\dots,c_n)$. □

Corollary 3.2. The variety of Heyting Wajsberg algebras (**HW**) has FMP.

Proof. By Theorem 3.7 and termwise equivalence between Stonean MV-algebras and Heyting Wajsberg algebras. □

Theorem 3.8. An algebra $\mathscr{A} \in$ **HW** is subdirectly irreducible if and only if it is a chain.

Proof. We prove separately the two implications.

($\Rightarrow$) For the left-to-right direction, by definition for every subdirect embedding $f : \mathscr{A} \mapsto \prod_{i\in I}\mathscr{A}_i$ there is an $i \in I$ such that $\pi_i \circ f : \mathscr{A} \mapsto \mathscr{A}_i$ is an isomorphism. Either by the subdirect representation Theorem for HW-algebras [21] or by the subdirect representation Theorem for Stonean MV-algebras [20] and termwise equivalence between Stonean MV-algebras and Heyting Wajsberg algebras for every HW-algebra $\mathscr{A}$ there is a subdirect embedding $h : \mathscr{A} \mapsto \prod_{i\in I}\mathscr{A}_i$ and for every $i \in I$, $\pi_i \circ f : \mathscr{A} \mapsto \mathscr{A}_i$ is an homomorphism and each $\mathscr{A}_i$ is a chain. Then $\mathscr{A}$ is isomorphic to a chain.

($\Leftarrow$) For the right-to-left direction, suppose $\mathscr{A}$ is a chain. By termwise equivalence between Stonean MV-algebras and Heyting Wajsberg algebras, $\mathscr{A}$ is termwise definitionally equivalent to a linear Stonean MV-algebra $\mathscr{A}'$. Since by linearity in $\mathscr{A}'$ we have $\sim=\sim_0$, there can be only two congruences-ideals: $\{\mathbf{0}\}$ and the whole A', that is to say $\mathscr{A}'$ is simple and hence so it is $\mathscr{A}$. By Theorem 8.1(II) in [19], $\mathscr{A}$ is simple if and only if $\mathscr{A}$ is subdirectly irreducible.

□

Lemma 3.2. For each n, $0 \neq n \in N$, any two n-element Heyting Wajsberg chains are isomorphic.

Proof. Trivial, by termwise equivalence with MV-chains that are Stonean by $\sim_0$. □

Combining Theorem 3.8 and Lemma 3.2 the following theorem yields:

Theorem 3.9. Up to isomorphism, there is precisely one subdirectly irreducible HW-algebra for each finite cardinality.

The lattice of subvarieties of most discriminator varieties is totally ordered. Nevertheless I am going to show this is not the case of Heyting Wajsberg algebras. Let us notice $\mathscr{A}_n$ the HW-algebra (or MV-algebra) of cardinality n.

Theorem 3.10. The lattice of subvarieties of Heyting Wajsberg algebras is not totally ordered. Moreover for any n, $2 < n \in N$, HSP($\mathscr{A}_n$) $\nsubseteq$ HSP($\mathscr{A}_{n+1}$) and HSP($\mathscr{A}_{n+1}$) $\nsubseteq$ HSP($\mathscr{A}_n$).

Proof. The lattice of subvarieties of MV-algebras is not totally ordered (see [17]). In the same book it is shown that in the case of MV-algebras for any couple of prime natural numbers p_i, p_j we have that HSP$(\mathscr{A}_{p_i+1}) \nsubseteq$ HSP$(\mathscr{A}_{p_j+1})$ and HSP$(\mathscr{A}_{p_j+1}) \nsubseteq$ HSP$(\mathscr{A}_{p_i+1})$. In [22] and [24] it is proved that any finite MV-algebra is isomorphic to a direct product of a family of linear MV-algebras. Any direct product of linear MV-algebras is a Stonean MV-algebra by the definition of Stonean negation $\sim_0$ componentwise. Then any finite MV-algebra is Stonean and termwise definitionally equivalent to a HW-algebra. Since the Stonean operator can't be defined as a combination of the MV-operators $\oplus$ and $\neg$, the class of theorems of a Stonean MV-algebra is the set-union of the set of theorems where the Stonean operator occurs and the set of theorems where it does not occur. These two sets are disjoint. Trivially an equation in the language of MV-algebras that is not satisfied in an MV-algebra continues not to be satisfied in the language of Stonean MV-algebra whereas the MV-algebra is Stonean. It follows that in the lattice of subvarieties of Stonean MV-algebras HSP$(\mathscr{A}_{p_j+1}) \nsubseteq$ HSP$(\mathscr{A}_{p_i+1})$ and HSP$(\mathscr{A}_{p_i+1}) \nsubseteq$ HSP$(\mathscr{A}_{p_j+1})$. By termwise equivalence in the lattice of subvarieties of Heyting Wajsberg algebras HSP$(\mathscr{A}_{p_i+1}) \nsubseteq$ HSP$(\mathscr{A}_{p_j+1})$ and HSP$(\mathscr{A}_{p_j+1}) \nsubseteq$ HSP$(\mathscr{A}_{p_i+1})$. For the sake of completeness we present a pair of equations in order to provide an infinite set of counterexamples to linearity. We leave to the reader the task to verify that for any integer $n > 1$,

$$\mathscr{A}_{n+2} \models (nx)^2 = (n-1)x$$
$$\mathscr{A}_{n+1} \not\models (nx)^2 = (n-1)x$$

and for any integer $n \geq 0$,

$$\mathscr{A}_{n+1} \models nx = (n+1)x$$
$$\mathscr{A}_{n+2} \not\models nx = (n+1)x$$

□

4. Logical results

Theorem 4.1. Let α be a formula of $Fm(\Lambda)$. Then, $\vdash_{GLL} \alpha$ iff α is a 1-tautology

Proof. We prove separately the two implications.

($\Rightarrow$) For the left-to-right direction, it can be easily verified that axioms (Ax1)-(Ax9) are 1-tautologies and that Modus Ponens (MP1), the only deduction rule of *GLL*, cannot decrease the evaluation of an inferred formula.

($\Leftarrow$) For the left-to-right direction, if α is a 1-tautology then the algebraic term related to α, meant in the natural traditional way (see p.21 in [13]), t_α implies $\mathscr{A}_{[0,1]} \models t_\alpha = 1$

By the standard algebraic completeness expressed in Theorem 2.3, if $t_\alpha = 1$ is satisfied in $[0,1]$ it is satisfied in any model of **HW** and thus $[\alpha]_\equiv$ is the top element of the Lindenbaum Tarski algebra of *GLL*. Hence $\vdash_{GLL} \alpha$. □

We have thus proved that *GLL* is the **1**-assertional logic of the variety **HW**, in Symbols $GLL = S(\mathbf{HW}, 1)$. Since **HW** is a **1**-regular variety, its **1**-assertional logic is regularly algebraisable with **HW** as equivalent algebraic semantics. A system of equivalence formulas is given by $\{\alpha \to_L \beta, \beta \to_L \alpha\}$.

Let $\Phi := \langle \to_L, \neg \rangle$ be a language of type $\langle 2,1 \rangle$. $\aleph_0$-valued Łukasiewicz logic, in symbols $Ł := \langle \Phi, \vdash_Ł \rangle$ is the deductive system presented by the following collection of axioms (Ł1-4) and inference rule (MP):

(Ł1) $\alpha \to_L (\beta \to_L \alpha)$
(Ł2) $(\alpha \to_L \beta) \to_L ((\beta \to_L \gamma) \to_L (\alpha \to_L \gamma))$
(Ł3) $((\alpha \to_L \beta) \to_L \beta) \to_L ((\beta \to_L \alpha) \to_L \alpha)$
(Ł4) $(\neg\beta \to_L \neg\alpha) \to_L (\alpha \to_L \beta)$
(MP) $\frac{\alpha, \alpha \to_L \beta}{\beta}$

It can be easily observed that the language Φ can be defined into the language Λ and thus the induced set of formulas $Fm(\Lambda)$ is an extension of $Fm(\Phi)$.

Theorem 4.2. For any formula ψ in the set of formulas defined by induction on the traditional way in Φ (i.e.$Fm(\Phi)$):

$$\vdash_{Ł} \psi \ \Rightarrow \vdash_{GLL} \psi$$

Proof. If $\vdash_{Ł} \psi$ by the semantical completeness theorem for the $\aleph_0$-valued propositional calculus proved by C. C. Chang (see [13]) and termwise equivalence between MV-algebras and Wajsberg algebras (also reported in [13]), let $\mathscr{W}_{[0,1]}$ be the standard real unit interval Wajsberg algebra, $\mathscr{W}_{[0,1]} \models \psi$. By Lemma 2.2 the Wajsberg algebra $\mathscr{W}_{[0,1]}$ is a term reduct of the standard real unit interval Heyting Wajsberg algebra $\mathscr{A}_{[0,1]}$, thus $\mathscr{A}_{[0,1]} \models \psi$. By Theorem 4.1, $\vdash_{GLL} \psi$. □

We introduce the intuitionistic propositional calculus IPC. Let $\Omega := \langle \wedge, \vee, \rightarrow_G, \mathbf{0} \rangle$ be a language of type $\langle 2,2,2,0 \rangle$. The intuitionistic propositional calculus, in symbols $IPC := \langle \Omega, \vdash_{IPC} \rangle$ is the deductive system presented by the following collection of axioms (I1-9) and inference rule MP:

(I1) $\alpha \rightarrow_G (\beta \rightarrow_G \alpha)$
(I2) $\alpha \rightarrow_G (\beta \rightarrow_G (\alpha \wedge \beta))$
(I3) $(\alpha \wedge \beta) \rightarrow_G \alpha$
(I4) $(\alpha \wedge \beta) \rightarrow_G \beta$
(I5) $\alpha \rightarrow_G (\alpha \vee \beta)$
(I6) $\beta \rightarrow_G (\alpha \vee \beta)$
(I7) $(\alpha \vee \beta) \rightarrow_G ((\alpha \rightarrow_G \gamma) \rightarrow_G ((\beta \rightarrow_G \gamma) \rightarrow_G \gamma))$
(I8) $(\alpha \rightarrow_G \beta) \rightarrow_G ((\alpha \rightarrow_G (\beta \rightarrow_G \gamma)) \rightarrow_G (\alpha \rightarrow_G \gamma))$
(MP) $\frac{\alpha, \alpha \rightarrow_G \beta}{\beta}$

It can be easily observed that the language Ω can be defined into the language Λ. Then the induced set of formulas $Fm(\Lambda)$ is an extension of $Fm(\Omega)$.

Theorem 4.3. For any formula ρ in the set of formulas defined by induction on the traditional way in Ω (i.e.$Fm(\Omega)$):

$$\vdash_{IPC} \rho \ \Rightarrow \vdash_{GLL} \rho$$

Proof. If $\vdash_{IPC} \rho$ by the completeness theorem of IPC respect to the class of Heyting algebras [15], let $\mathscr{H}_{[0,1]}$ be the standard real unit interval Heyting algebra, $\mathscr{H}_{[0,1]} \models \rho$. By Lemma 2.2 the Heyting algebra $\mathscr{H}_{[0,1]}$ is a term reduct of the standard real unit interval Heyting Wajsberg algebra $\mathscr{A}_{[0,1]}$, thus $\mathscr{A}_{[0,1]} \models \rho$. By Theorem 4.1, $\vdash_{GLL} \rho$. □

Definition 4.1. A deductive system S over a pointed language type with distinguished constant symbol $\mathbf{1}$ is said to be a *pointed discriminator logic* if it arises as the $\mathbf{1}$-assertional logic of a pointed discriminator variety.

Corollary 4.1. Gödel Łukasiewicz Logic is a pointed discriminator logic.

Definition 4.2. Let S be a deductive system over a language type Λ. A nonempty set $\{\sigma_i \mid i \in I\}$ of binary Λ-formulas is said to be a *deduction-detachment system* for S if, for all $\Gamma \cup \{\alpha, \beta\} \subseteq Fm(\Lambda)$ and for all $i \in I$:

$$\Gamma, \alpha \vdash_S \beta \quad \Leftrightarrow \quad \Gamma \vdash_S \sigma_i(\alpha, \beta).$$

S is said to have the *(uniterm) deduction-detachment theorem* (DDT) if it has a (unitary) deduction-detachment system.

DDT has been extensively studied in abstract algebraic logic. For a survey, see the tutorial paper [7].

Theorem 4.4. Gödel Łukasiewicz Logic has DDT.

Proof. We have proved the variety of Heyting Wajsberg algebras to be a discriminator variety and to have EDPC. W. Block and D. Pigozzi in [5] have proved that in such conditions the related arising $\mathbf{1}$-assertional logic (i.e. Gödel Łukasiewicz Logic) has DDT. □

Theorem 4.5. Gödel Łukasiewicz Logic is decidable.

Proof. In Corollary 3.2 we have proved the variety of Heyting Wajsberg algebras to have the finite model property (FMP). It is well known in abstract algebraic logics that if a finitely axiomatized variety has FMP then its $\mathbf{1}$-assertional logic is decidable. Thus *GLL* is decidable. □

Theorem 4.6. Gödel Łukasiewicz Logic is strongly complete.

Proof. In Theorem 3.4 we have proved the variety of Heyting Wajsberg algebras to be complete with respect to the standard unitary real interval model as quasivariety (i.e. with respect to quasi-equations). Then its $\mathbf{1}$-assertional logic (i.e. *GLL*) is strongly complete. □

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