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A New Conception of Equality of Tautologies

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ABSTRACT. It is well-known that there are "hard" and "simple" tautologies, but in the capacity of the logical functions they all are equal to each other. In our opinion this thesis is not entirely correct. We suggest a new conception of equality of tautologies, with the help of the notion of φ-determinative conjunct, which was defined in [1] for every tautology φ.

KEYWORDS: φ-determinative conjunct, minimal determinative disjunctive normal form, equality of tautologies.

1. Introduction

In this paper we would like to discuss a conceptual question: in what case two tautologies can be considered as equal. Let φ and ψ be propositional formulae (logical functions) and let each of them depend on the propositional variables $p_1, p_2, \dots, p_n$. It is well-known that φ and ψ are equal iff for every $\sigma = (\sigma_1, \dots, \sigma_n)$ ($\sigma_i \in \{0,1\}$, $1 \leqslant i \leqslant n$) $\varphi(\sigma_1, \dots, \sigma_n) = \psi(\sigma_1, \dots, \sigma_n)$. By this conception all tautologies are equal to each other. In our opinion this thesis is not entirely correct.

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In fact, the tautology $\varphi_k = (p_1 \supset (p_2 \supset (p_3 \supset \ldots \supset (p_k \supset p_1)\ldots)))$ is very "simple". It is easy to notice that (i) if the value of p_1 is 1, then, because of its second occurrence, the value of φ is equal to 1 without taking into consideration the values of the remaining variables, and (ii) if the value of p_1 is 0 then the value of φ is 1 because of the first occurrence of p_1. So, only the variable p_1 is "important" in this formula, while the other variables are absolutely unimportant. In some tautologies several variables are "important", and there are also tautologies where nearly all variables are "important". It is natural that such tautologies are "harder" (some examples of "hard" tautologies will be adduced below). In [1] the notion of φ-determinative conjunct was defined. Using this notion we suggest a new definition of equality of tautologies according to which two tautologies can be considered as equal iff they have the same "hardness".

2. Preliminary

This paper deals exclusively with classical propositional logic. We shall use the generally accepted concepts of unit Boolean cube (E^n), logical function, propositional formula, tautology, conjunct, and disjunctive normal form (DNF).

The particular choice of language for the representation of propositional formulae does not matter for our analysis. However, because of technical reasons we assume that our language contains the propositional variables p_i ($i \geqslant 1$), the logical symbols $\neg$, &, $\vee$, $\supset$ and the parentheses (,). Note that some of the parentheses can be disregarded in generally accepted cases.

It is well-known that every propositional formula is a presentation of a specific logical function and every logical function can be represented by means of different propositional formulae and, particularly, by different DNFs.

In some cases we shall identify the propositional formula with the logical function, which is presented by this formula.

Two given formulae $\varphi(p_1, p_1, \ldots, p_n)$ and $\psi(p_1, p_1, \ldots, p_n)$ are considered as equal, according to the usual terminology, iff they present the same logical function, i.e. for each $\sigma^n = (\sigma_1, \ldots, \sigma_n) \in E^n$ $\varphi(\sigma_1, \ldots, \sigma_n) = \psi(\sigma_1, \ldots, \sigma_n)$.

We call *replacement-rule* the following trivial equalities for each propositional formula A:

$$0\&A=0,\ A\&0=0,\ 1\&A=A,\ A\&1=A,\ A\&A=A,\ A\&\bar{A}=0,\ \bar{A}\&A=0$$
$$0\vee A=A,\ A\vee 0=A,\ 1\vee A=1,\ A\vee 1=1,\ A\vee A=A,\ A\vee\bar{A}=1,\ \bar{A}\vee A=1$$
$$0\supset A=1,\ A\supset 0=\bar{A},\ 1\supset A=A,\ A\supset 1=1,\ A\supset A=1,\ A\supset\bar{A}=\bar{A},\ \bar{A}\supset A=A$$
$$\bar{0}=1,\ \bar{1}=0,\ \bar{\bar{A}}=A.$$

Let φ be a propositional formula and let $\{p_1,p_2,\dots,p_n\}$ be the set of its distinct variables. For some $\sigma^m=(\sigma_1,\dots,\sigma_m)\in E^m$ $(1\leqslant m\leqslant n)$ the conjunct $K=p_{i_1}^{\sigma_1}\&p_{i_2}^{\sigma_2}\&\dots\&p_{i_m}^{\sigma_m}$ is called *φ-determinative* if the assignment of values σ_j to each p_{i_j} $(1\leqslant j\leqslant m)$ induces the value (1 or 0) for φ, without taking into consideration the values of the remaining variables [1], i.e. if the value for φ is obtained using the above replacement-rule after the assignment of the value σ_j to p_{ij}. It is obvious that for $m=n$ every conjunct $K=p_{i_1}^{\sigma_1}\&p_{i_2}^{\sigma_2}\&\dots\&p_{i_n}^{\sigma_n}$ for each $\sigma^n=(\sigma_1,\dots,\sigma_n)\in E^n$ is φ-determinative. The case for $m<n$ is more interesting.

EXAMPLE
Let $\varphi=(p_1\&p_2)\supset(p_3\vee(\bar{p}_4\&p_5))$.
It is easy to check that the following conjuncts are φ-determinative:

$$\mathscr{K}_1=p_1^0,\mathscr{K}_2=p_2^0,\mathscr{K}_3=p_3^1,\mathscr{K}_4=p_4^0\&p_5^1,\mathscr{K}_5=p_1^0\&p_2^1\&p_3^0\&p_4^1\&p_5^1$$

but the conjuncts $\mathscr{K}_6=p_1^1\&p_3^0$, $\mathscr{K}_7=p_3^0\&p_5^1$, $\mathscr{K}_8=p_2^1$ are not φ-determinative.

It is well-known that every logical function φ can be represented by different DNFs: minimal, short, dead etc. In each of these DNFs some parameter (number of occurrences of variables, number of conjuncts, etc.) has its minimal value. It is important to note that *every conjunct from every* DNF *φ is φ-determinative.*

3. Notion of $\mathscr{D}_\varphi^{min}$. Equality of tautologies

According to the traditional view of equality of propositional formulae, mentioned at the beginning of this paper, all tautologies must be equal to each other. In our opinion this thesis is not entirely correct. Here we suggest a new conception of equality of tautologies based on the notion of φ-determinative conjunct.

In the ordinary terminology we call variables and negated variables *literals*; the conjunct $\mathscr{K}$ can be represented simply as the sets of literals and is called *clause* (no clause contains both a variable and the negation of that variable). A formula in DNF can be expressed as a set of clauses $\{\mathscr{K}_1, \mathscr{K}_2, \ldots, \mathscr{K}_\ell\}$.

The *elimination-rule* (ε-rule) infers $\mathscr{K}' \cup \mathscr{K}''$ from clauses $\mathscr{K}' \cup \{p\}$ and $\mathscr{K}'' \cup \{\bar{p}\}$, where $\mathscr{K}'$ and $\mathscr{K}''$ are clauses and p is a propositional variable.

We would like to say that the conjunct K is deduced from the DNF $\mathscr{D}$ if there is a finite sequence of clauses such that every clause in the sequence is one of the clauses of $\mathscr{D}$ or is inferred from earlier clauses in the sequence by ε-rule, and the last clause is $\mathscr{K}$.

DNF $\mathscr{D}$ is called *full* (tautology) if the empty conjunct (Λ) can be deduced from $\mathscr{D}$.

The minimal number of the usages of ε-rule in the deduction of Λ from full DNF $\mathscr{D}$ is called *complexity* of $\mathscr{D}$ and denoted by $C(\mathscr{D})$.

Let φ be some tautology.

A full DNF $\mathscr{D}$ is called φ-*determinative* if every conjunction of $\mathscr{D}$ is φ-determinative. Any φ-determinative DNF $\mathscr{D}$ with minimal complexity is called *minimal determinative* DNF for φ and denoted by $\mathscr{D}_\varphi^{min}$. It is natural to take the value of $C\left(\mathscr{D}_\varphi^{min}\right)$ as characterizing the complexity of validity of the formula φ.

EXAMPLES

1. Let $\alpha_k = p_1 \supset (p_2 \supset (\ldots \supset (p_{k-1} \supset (p_k \supset (\bar{p}_k \supset p_1)))\ldots))$, with $k \geqslant 2$.

   It is easy to check that the following DNF are α_k-determinative:

   $$\mathscr{D}_1 = \{p_1; \bar{p}_1\}, \quad \mathscr{D}_2 = \{p_k; \bar{p}_k\}, \quad \mathscr{D}_3 = \{p_1 p_2; p_1 \bar{p}_2; \bar{p}_1\},$$

   but only $\mathscr{D}_1$ and $\mathscr{D}_2$ are minimal determinative for α_k and $C(\mathscr{D}_1) = C(\mathscr{D}_2) = 1$.

2. Let

   $$\begin{aligned} \beta_\ell &= (p_1 \supset p_2) \supset ((p_2 \supset p_3) \supset \\ &\quad (\ldots \supset ((p_{\ell-1} \supset p_\ell) \supset (p_1 \supset p_\ell))\ldots)) \end{aligned}$$

   with $\ell \geqslant 3$.

   $\mathscr{D}_{\beta_\ell}^{min} = \{p_1\bar{p}_2; p_2\bar{p}_3; \ldots; p_{\ell-1}\bar{p}_\ell; \bar{p}_1; p_\ell\}$ and $C\left(\mathscr{D}_{\beta_\ell}^{min}\right) = \ell$.

Note that the problem of the construction of $\mathscr{D}_{\varphi}^{min}$ is unfortunately NP-hard (this follows from the results proved in [2] and [3]).

Let the size (the number of all symbols) of a formula φ be denoted by $|\varphi|$.

The following statements about $\mathscr{D}_{\varphi}^{min}$ are proved in [1] and [2]:

1. If every φ-determinative conjunct contains at least m literals for any tautology φ, then $C\left(\mathscr{D}_{\varphi}^{min}\right) \geqslant 2^m$.

2. For every tautology φ of size n, $C\left(\mathscr{D}_{\varphi}^{min}\right) \leqslant 2^n$.

3. For sufficiently large n, sequences of tautologies φ_n of size n are described, such that $C\left(\mathscr{D}_{\varphi_n}^{min}\right)$ are of order $n, n^2, n^3, \ldots, n^{\left[\frac{n}{2}\log_n 2\right]}$.

PROOF SKETCH

1. If every φ-determinative conjunct contains at least m literals, then every φ-determinative DNF $\mathscr{D}$ must contain at least 2^m conjuncts.
2. It is obvious that if φ is a tautology of size n, then the number of the distinct variables of φ is less than n, and if full $\mathscr{D}$ is the canonic DNF, i.e. every clause of $\mathscr{D}$ contains the literals of all variables, then $C(\mathscr{D}) \leqslant 2^n$, hence $C\left(\mathscr{D}_{\varphi}^{min}\right)$ is also less than 2^n.
3. Let $\varphi_{s,m}$ be the formula $\bigvee_{(\sigma_1 \ldots \sigma_s) \in E^s} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{s} p_{ij}^{\sigma_i}$. It is not difficult to notice that $\varphi_{s,m}$ are tautologies for every $s \geqslant 1$ and $1 \leqslant m \leqslant 2^s - 1$, therefore the tautologies

$$\varphi_{s,1}, \varphi_{s,s}, \varphi_{s,s^2}, \varphi_{s,s^3}, \ldots, \varphi_{s,s^{[(s-1)\log_s 2]}}$$

can be considered to form a sequence of the kind described above.

The notion of $C\left(\mathscr{D}_{\varphi}^{min}\right)$ is useful also for the evaluation of proof complexity. In particular, this notion is used in [1] and [2] in order to prove the following results:

1. For sufficiently large n, there are sequences of tautologies φ_n of size n such that their proof-complexities (the steps of proof and/or the size of proof) in "weak" proof systems of classical propositional logic (like resolution system, cut-free sequent system) are of order $n, n^2, n^3, \ldots, n^{\left[\frac{n}{2}\log_n 2\right]}$.

2. Every tautology φ can be proved inside a Frege system (the most natural calculus for propositional logic) in less than $c_1 \cdot C\left(\mathscr{D}_{\varphi}^{min}\right)|\varphi|$ steps and in less than $c_2 \cdot C\left(\mathscr{D}_{\varphi}^{min}\right)|\varphi|^2$ size, where c_1 and c_2 are constants, and therefore, as above, for sufficiently large n, there is the sequence of such tautologies φ_n of size n, for which the upper bounds of Frege proof complexities are of order $n, n^2, n^3, \ldots, n^{\left[\frac{n}{2}\log_n 2\right]}$.

All the above results suggest that the value of $C\left(\mathscr{D}_{\varphi}^{min}\right)$ is important for the validity (derivability) of a tautology φ.

So, we can notice that $C(\mathscr{D})$ may be "small" (as in the case of α_k and β_ℓ in Examples 1. and 2.) and can be "large" (for $\varphi_{s,2^s-1}$). We can choose $k = \ell = s(2^s - 1)$ so that the number of variables of tautologies α_k, β_ℓ and $\varphi_{s,2^s-1}$ will be equal to each other, but α_k, β_ℓ are "simple" and $\varphi_{s,2^s-1}$ is very "hard".

Taking into consideration the above-mentioned arguments, we suggest the following definition of equality of tautologies:

Definition. The tautologies φ and ψ are strongly equal if every φ-determinative conjunct is also ψ-determinative and vice versa.

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