L&PS – Logic & Philosophy of Science
Vol. III, No. 1, 2005

Many-valued Logics Enriched with a Stonean Negation: a Direct Proof of Representation and Completeness

Martinvaldo Konig
University of Cagliari, Italy
e-mail: martinvk@iol.it

1. Basic notions
2. Stonean Negation and Stonean MV-algebras
3. Subdirect Representation
4. Algebraic Completeness

ABSTRACT. This paper studies Łukasiewicz's many-valued logic enriched with a new operator: the Stonean negation. This research focuses on the class of algebras containing the algebraic counterpart of this new logic: the class of Stonean MV-algebras. A direct proof of subdirect representation Theorem is given, as well as an algebraic completeness Theorem.

KEYWORDS. MV-algebra, Stonean MV-algebra, Stonean negation operator, Chang's subdirect representation, Chang's algebraic completeness.

Introduction

There is a direct relationship between any logical calculus S and the class of adequate models for it - i.e. the class of algebraic structures which verify exactly the provable formulae of S. For example Boolean algebras are the algebraic counterpart of classical propositional logic and Heyting algebras correspond to intuitionistic propositional logic (see pp.380-3 in [DH01]). This fruitful interaction allows algebraic investigation to have a direct insight into a given calculus and conversely pure proof-theoretical techniques may contribute to pursue algebraic results.

The concept of MV-algebra was first introduced by C.C.Chang [Ch58] to provide a new proof of completeness of Łukasiewicz's $\aleph_0$-valued propositional calculus

[CDM00]. His proof [Ch59], however, is not self-contained, since it exploits the completeness of the first order language of totally ordered Abelian group theory.

In Chang's analysis the model based on the unit interval of real numbers [0,1] is prototypical, because it represents the set of truth-values image of the evaluation map of Łukasiewicz's $\aleph_0$-valued propositional calculus.

Chang [Ch58] defined an algebra with two primitive operators. Their mutual combinations allowed him to define new useful operators as well as their dual ones.

Recently MV-algebras enriched with new operators has been widely investigated. These new operators have an intended clear interpretations in [0,1]-model (i.e. the so called standard MV-algebra).

One of these operators is the Stonean negation studied by L.P.Belluce in [Be97]. It should be mentioned that a similar negation operator (i.e. Δ-operator) had been presented by Matthias Baaz in [Ba96]. In this article Baaz studied Gödel's infinite-valued logic [Go33] enriched with Δ-operator. In a linear structure with involution (for instance in standard MV-algebra) Baaz's Δ-operator and Stonean negation are trivially mutually definable.

Later Petr Hajek in [Ha98] utilized Baaz's Δ-operator to enrich Łukasiewicz's many-valued logic. The axiomatization of this new logical system is provided by traditional Łukasiewicz's axioms, an added set of five axioms (Δ1-Δ5), Modus Ponens and a new rule of Generalization. The Lindenbaum-Tarski [CDM00] algebra of this new logic gives rise to a new class of algebraic structure: MVΔ-algebras. These structures are particular MV-algebras whose axioms are Łukasiewicz's ones together with a set of six equations, which are the algebraic counterparts of (Δ1-Δ5) and of Generalization rule. P.Hajek [Ha98] proved a subdirect representation Theorem as well as a completeness Theorem for this new logic.

F.Esteva, L.Godo and F.Montagna [EGM01] extended Hajek's system and its results into a generalized logical system (LΠ) containing Łukasiewicz's logic, Product logic [EGH96], Gödel's infinte-valued logic, Takeuti and Titani's propositional logic [TT92], Pavelka's rational and product logics [Pa79], Łukasiewicz's logic with Δ, Product and Gödel's logics with Δ and involution [EGHN00].

Some connections between MVΔ-algebras and Stonean MV-algebras suggest that representation and completeness for Stonean MV-algebras could be obtained in a indirect way from Hajek's results.

The main aim of this paper is to give a direct proof of these results, remaining in its natural algebraic approach.

The Stonean property is what characterizes exactly at the algebraic level this whole class of MV-algebras.

In [0,1]-model Stonean negation η and involution $\neg$ allow the easiest definitions of the modal operators of necessity $\Box$ and possibility $\Diamond$ by their mutual combination (in fact $\Box x = \eta\neg x$ and $\Diamond x = \neg\eta x$).

Furthermore G.Cattaneo, R.Giuntini and R.Pilla [CGP98] have proved that Stonean MV-algebras are equivalent to BZMV^{dM}-algebras and hence, thanks to the completeness of the logic based on Stonean MV-algebras we prove in this paper, the completeness of the one based on BZMV^{dM}-algebras follows. Furthermore the proof of the subdirect representation Theorem given in [CGP98] is incomplete, then we provide a correct proof of this result for BZMV^{dM}-algebras.

Since this work is purely algebraic we leave to the readers the chance of providing a proper logical sintax and an adequate axiomatization.

1. Basic notions

In this section we present MV-algebra axioms and some basic notions and recalls. The axiomatization we are going to introduce is mainly due to Mangani [Ma73]. It provides only five axioms while Chang's original one [Ch58] was determined by eleven equations and their dual versions.

Definition 1.1 An *MV-algebra* is a structure $\mathcal{A}$=$< A, \oplus, \neg, \mathbf{0} >$ of type $< 2, 1, 0 >$. For any $x, y \in A : x \vee y \stackrel{\text{def}}{=} \neg(\neg x \oplus y) \oplus y$. The following axioms are required:

(MV1) $(x \oplus y) \oplus z = (y \oplus z) \oplus x$

(MV2) $x \oplus \mathbf{0} = x$

(MV3) $x \oplus \neg\mathbf{0} = \neg\mathbf{0}$

(MV4) $x \vee y = y \vee x$

(MV5) $\neg\neg x = x$

It is useful to define also the dual concepts: $\mathbf{1} \stackrel{\text{def}}{=} \neg\mathbf{0}$, $x \odot y \stackrel{\text{def}}{=} \neg(\neg x \oplus \neg y)$, and $x \wedge y \stackrel{\text{def}}{=} \neg(\neg x \odot y) \odot y$.

We observe that the relation $x \leq y \Leftrightarrow x \vee y = y$ induces in every MV-algebra a distributive lattice order.

An MV-algebra $\mathcal{A}$ is *linear (or totally ordered)* iff for any pair of elements $x, y \in A$, either $x \leq y$ or $y \leq x$.

Example 1.1. *Standard MV-algebra.*

$\mathcal{A}_{[0,1]} =< [0, 1], \oplus, \neg, 0 >$

where $[0, 1] \subset R$,

$x \oplus y \stackrel{\text{def}}{=} min\{1, x + y\}$

and $\neg x \stackrel{\text{def}}{=} 1 - x$.

In the sequel we'll adopt the following notation. Given an MV-algebra $\mathcal{A}$,$\forall x \in A$ and $\forall n \in N$:

$$nx = \begin{cases} \mathbf{0} & \text{if } n = 0 \\ x & \text{if } n = 1 \\ \underbrace{x \oplus \ldots \oplus x}_{n-times}, & \text{if } 2 \leq n \in N \end{cases}$$

Definition 1.2. Let $\mathcal{A}$ be an MV-algebra and $x \in A$. The *order of* x is the smallest $n \in N$ if it exists s.t. $nx = \mathbf{1}$. If n does not exist we say that the order of x is infinite.

Definition 1.3. An MV-algebra $\mathcal{A}$ is *archimedean* if and only if $\forall x \in A$ and $\forall n \in N, nx \leq y \Rightarrow x \odot y = x$.

It can easily observed that the standard MV-algebra of example 1.1 is linear and archimedean (i.e. $\forall x \neq \mathbf{0}$, x has a finite order) meanwhile Chang's MV-algebra $\mathcal{C}$ ([Ch58], p.474) provides the smallest example of linear but not archimedean MV-algebra.

Definition 1.4. An *ideal* J of an MV-algebra $\mathcal{A}$ is a subset of A which satisfies the following conditions:
(I1) $\mathbf{0} \in J$
(I2) if $x \in J$ and $y \leq x$, then $y \in J$
(I3) if $x \in J$ and $y \in J$, then $x \oplus y \in J$

Definition 1.5. An ideal J of an MV-algebra $\mathcal{A}$ is *proper* iff $\mathbf{1} \notin J$.

Definition 1.6. Let $\mathcal{A}$=$< A, \oplus, \neg, \mathbf{0} >$ be an MV-algebra, let I be an ideal of $\mathcal{A}$ and $x \in A$. We introduce the definition of *ideal generated by* $I \cup \{x\}$, denoted $Id(I \cup \{x\})$:
$Id(I \cup \{x\}) \stackrel{\text{def}}{=} \{y \in A \mid y \leq i \oplus nx$, for some $i \in I$ and some $n \in N\}$.
Further the *ideal generated by* x:
$Id(x) \stackrel{\text{def}}{=}$ the ideal generated by $\{\mathbf{0}\} \cup \{x\}$.

Definition 1.7. An ideal J of an MV-algebra $\mathcal{A}$ is *maximal* iff it is proper and for any ideal I of $\mathcal{A}$ s.t. $J \subseteq I$, either $I = J$ or $I = A$.

Definition 1.8. An ideal J of an MV-algebra $\mathcal{A}$ is *prime* iff it is proper and if for any pair of elements $x, y \in A$, either $x \odot \neg y \in J$ or $\neg x \odot y \in J$.

Definition 1.9. *Distance function on an MV-algebra* $\mathcal{A}$.
Let $x, y \in A$, $d(x, y) \stackrel{\text{def}}{=} (\neg x \odot y) \oplus (x \odot \neg y)$.

Definition 1.10. Let J be an ideal of an MV-algebra $\mathcal{A}$, $\forall x, y \in A$:
$x \equiv_J y \Leftrightarrow d(x, y) \in J$.

C.C.Chang [Ch58] had shown that the above relation $\equiv_J$ is reflexive, symmetric and transitive. Moreover in any MV-algebra $\mathcal{A}$, $\forall x, y, w, z \in A$ and for any ideal J of $\mathcal{A}$, the following conditions hold:
1) $x \equiv_J y$ and $w \equiv_J z \Rightarrow x \oplus w \equiv_J y \oplus z$,
2) $x \equiv_J y \Rightarrow \neg x \equiv_J \neg y$.
Thus $\equiv_J$ is a congruence relation and in the same way of group theory [He82] it induces a quotient MV-algebra $\mathcal{A}/J$ homomorphic to the original $\mathcal{A}$. Moreover C.C.Chang proved [Ch58] that if J is prime, then the quotient MV-algebras $\mathcal{A}/J$ is linear . Let us now define the last main concepts necessary to present Chang's representation Theorem.

Definition 1.11. A *direct product* of a given family of MV-algebras $\{\mathcal{A}_i \mid i \in I\}$ is an MV-algebra $\Pi_{i\in I}\mathcal{A}_i$=$<\ \Pi_{i\in I}A_i, \oplus, \neg, \mathbf{0}\ >$ where $\Pi_{i\in I}A_i$ = the cartesian product of $\{A_i \mid i \in I\}$ and the operators are defined componentwise as the operators of each original MV-algebra $\mathcal{A}_i$. The $\mathbf{0}$-element is obviously the sequence of all the $\mathbf{0}$-elements of $\{A_i \mid i \in I\}$.

Every element x of a direct product $\Pi_{i\in I}\mathcal{A}_i$ of MV-algebras $\{\mathcal{A}_i \mid i \in I\}$ is expressed in the following way: $x =< x_1, ..., x_n, .. >$ where each x_i belongs to each MV-algebra $\mathcal{A}_i$ of $\Pi_{i\in I}\mathcal{A}_i$. We will use often this notation.

Definition 1.12. Let an MV-algebra $\Pi_{i\in I}\mathcal{A}_i$ be a direct product of a family of MV-algebras $\{\mathcal{A}_i \mid i \in I\}$ and $j \in I$.
Let $\pi_j : \Pi_{i\in I}A_i \mapsto A_j$ be the *j-th projection function* s.t. $\forall x =< x_1, ..., x_n, .. >\in \Pi_{i\in I}A_i$, $\pi_j(x) \stackrel{\text{def}}{=} x_j$.
An MV-algebra $\mathcal{A}$ is a *subdirect product* of a given family of MV-algebras $\{\mathcal{A}_i \mid i \in I\}$ iff there exists a one-one homomorphism $h : \mathcal{A} \mapsto \Pi_{i\in I}\mathcal{A}_i$ such that for any $j \in I$, the compose map $\pi_j \circ h$ is a homomorphism onto $\mathcal{A}_j$.

Obviously every subdirect product of a family of MV-algebras $\{\mathcal{A}_i \mid i \in I\}$ is a subalgebra of the direct product of the same family of MV-algebras.

C.C.Chang in his article [Ch59] has proved:

Chang's representation Theorem. Every MV-algebra is isomorphic to a subdirect product of linear MV-algebras.

We have already recalled how $\equiv_J$ is a congruence relation with respect to the basic operations of any MV-algebra (see also either [CDM94] or [CDM00]). In Chapter 3 we will prove that with an enriched definition of prime ideal I, $\equiv_I$ continues to be a congruence relation even respect to the new added operator we are going to introduce.

2. Stonean Negation and Stonean MV-algebras

Inspired by Ovchinnikov [Ov83] and by Belluce [Be97] we have begun to consider generalized operators on MV-algebras. Moreover we have focused our attention on negations and particularly on Stonean ones. These negations characterize a whole class of MV-algebras.

Definition 2.1. *The set of Boolean elements*
Let $\mathcal{A}$ be an MV-algebra. The set of all the idempotent elements of A is called the set of Boolean elements of A and it is denoted by $B(A)$.
$B(A) \stackrel{\text{def}}{=} \{x \in A \mid x \oplus x = x\}$

In [CDM00] it is proved that in any MV-algebra $\mathcal{A}$ the following properties hold:

(B1) $x \in B(A) \Rightarrow \forall y \in A, x \odot y = x \wedge y$
(B2) $x \in B(A) \Rightarrow \forall y \in A, x \oplus y = x \vee y$

Definition 2.2. Let $\mathcal{A}$ be an MV-algebra. A *negation operator* on $\mathcal{A}$ is a 1-ary function $\sim: A \mapsto A$ such that $\forall x, y \in A$:

(i) if $x \in B(A)$, then $\sim x \stackrel{\text{def}}{=} \neg x$ (this condition guarantees that $\sim$ is an extension of classical negation).
(ii) $x \leq y \Rightarrow \sim y \leq \sim x$.

Definition 2.3. An MV-algebra $\mathcal{A}$ is *Stonean* iff $\forall x \in A, \exists \eta(x) \in B(A)$, s.t. $\{y \mid y \wedge x = \mathbf{0}\} = \{y \mid y \leq \eta(x)\}$.

It can be easily observed that for any $x \in A$, the corresponding Boolean element $\eta(x)$ of the above definition is trivially unique.

Then a Stonean MV-algebra defines immediately a new operator:

Definition 2.4. Let $\mathcal{A}$ be a Stonean MV-algebra. A *Stonean* negation operator is a map η: $A \mapsto B(A)$ associating to any element $x \in A$ the unique above element $\eta(x)$.

Every linear MV-algebra is trivially Stonean once defined a Stonean negation operator as:

$$\eta_0(x) \stackrel{\text{def}}{=} \begin{cases} \mathbf{1} & \text{if } x = \mathbf{0} \\ \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Consequently every direct product of linear MV-algebras is Stonean with $\eta(x) \stackrel{\text{def}}{=} < \eta_0(x_1), ..., \eta_0(x_n), .. >$.

We provide now the smallest example of not Stonean MV-algebra: a subdirect product of two Chang's MV-algebras $\mathcal{C}$ ([Ch58], p.474) in which every element is a ordered pair of components that both belong either to the subset $\{0, c, c+c, c+c+c, ..\}$ of the support C or to the subset $\{1, 1-c, 1-c-c, 1-c-c-c, ..\}$ of the support C of $\mathcal{C}$. In this MV-algebra the set of the Boolean elements is composed only by the two elements $< 0, 0 >$ and $< 1, 1 >$. If $x =< c, 0 >$, $< 1, 1 >\notin \{y \mid y \wedge x = 0\}$, $< 0, 0 >\neq< 0, c >$ and $< 0, 0 >\leq< 0, c >\in \{y \mid y \wedge x = 0\}$. It proves this MV-algebra is not Stonean.

As shown by L.P.Belluce [Be97], in any Stonean MV-algebra $\mathcal{A}$, $\forall x, y \in A$ the following properties hold:

(P1) $x \wedge \eta x = \mathbf{0}$
(P2) $\eta(x \wedge y) = \eta x \vee \eta y$
(P3) $\eta(x \vee y) = \eta x \wedge \eta y$
(P4) $\eta x \oplus \eta x = \eta x$
(P5) $\eta\eta(x \oplus y) = \eta(\eta x \odot \eta y) = \eta\eta x \oplus \eta\eta y$
(P6) $\eta\neg x \leq x \leq \eta\eta x$

3. Subdirect Representation

We are going to deal with representation concerning Stonean MV-algebras. We have already recalled that every MV-algebra is isomorphic to a subdirect product of linear MV-algebras that are trivially Stonean. In a subdirect product of linear stuctures it is important to prove that the Stonean negation operator η of a given Stonean MV-algebra $\mathcal{A}$ is preserved in each component of its subdirect representation such that $\forall <x_1,...,x_n,..> \in A$, $\eta <x_1,...,x_n,..> = <\eta_0(x_1),...,\eta_0(x_n),..>$, since η_0 is the Stonean negation operator proper of any linear MV-algebra. To pursue this goal we need an adequate notion of ideal. We will utilize a definition first introduced by G.Cattaneo, R.Giuntini and R.Pilla in [CGP98], and their contribution into the analisys of BZMV^{dM}-algebras, although their proof of the representation Theorem of this structure is incomplete.

Definition 3.1. Let J be an ideal of a Stonean MV-algebra $\mathcal{A}$ with its Stonean negation operator η. J is a *Stonean ideal (s-ideal)* if and only if $\forall x, y \in A, x \odot y \in J \Rightarrow \eta\eta x \odot \eta\neg y \in J$.

Lemma 3.1. Let $\mathcal{A}$ be a Stonean MV-algebra, $\{\mathbf{0}\}$ is a s-ideal of $\mathcal{A}$.

Proof:

Let $x, y \in A$ and $x \odot y = \mathbf{0}$. Then $\neg y \oplus (x \odot y) = \neg y$ that is by definition $\neg y \vee x = \neg y$ and thus $\eta(\neg y \vee x) = \eta\neg y$. By (P3) we obtain $\eta\neg y = \eta\neg y \wedge \eta x$ and hence $\eta\eta x \odot (\eta\neg y \wedge \eta x) = \eta\eta x \odot \eta\neg y$. But since for (P1) $x \odot \eta x \leq x \wedge \eta x = \mathbf{0}$ and since $\eta x \in B(A)$ implies $\eta\eta x = \neg\eta x$ we have, by definition of $\wedge$ in terms of $\oplus$ and $\odot$, $\eta\eta x \odot (\eta\neg y \wedge \eta x) = \eta\eta x \odot \eta x \odot (\eta\neg y \oplus \eta\eta x) = \mathbf{0}$ and then, by substitution $\eta\eta x \odot \eta\neg y = \mathbf{0}$.

In a direct product of two standard MV-algebras of example 1.1 for $x = <1, .5>$ and $y = <.5, 1>$, $x \odot \neg y \neq \mathbf{0}$ and $\neg x \odot y \neq \mathbf{0}$. Hence in general $\{\mathbf{0}\}$ is a not prime s-ideal.

In a not archimedean linear MV-algebra $\mathcal{A}$, for instance in Chang's MV-algebra $\mathcal{C}$ ([Ch58], p.474), $\forall x \in A$, if $\mathbf{1} \neq x = na$, with $\mathbf{0} \neq a \in A$ and $1 < n \in N$, we have $x \odot 1 = x \in Id(x)$ and $\eta\eta x \odot \eta\neg 1 = 1 \notin Id(x)$. We recall that in every linear MV-algebra any ideal is trivially prime. Then here $Id(x)$ is a prime not Stonean ideal. Therefore in general neither the set of prime ideals is a subset of the set of s-ideals nor vice versa.

The relation $\equiv_J$ divides an MV-algebra into quotient MV-algebras related to its ideals. Moreover if these ideals are prime, their quotient MV-algebras are linear [Ch58]. $\equiv_J$ is a congruence relation with respect to the basic operations of any MV-algebra (see also either [CDM94] or [CDM00]). We have to prove that it also preserves the Stonean negation operator.

Theorem 3.1. Let $\mathcal{A}$ be a Stonean MV-algebra and let J be an s-ideal of $\mathcal{A}$, $\forall x, y \in A \quad x \equiv_J y \Rightarrow \eta x \equiv_J \eta y$.

Proof:
$x \equiv_J y \Leftrightarrow (x \odot \neg y) \oplus (\neg x \odot y) \in J$. Hence $(x \odot \neg y) \in J$ and $(\neg x \odot y) \in J$. It implies, being J an s-ideal, that $\eta\eta x \odot \eta\neg\neg y = \eta\eta x \odot \eta y \in J$ and $\eta\eta y \odot \eta\neg\neg x = \eta\eta y \odot \eta x \in J$. This is equivalent to $\eta x \equiv_J \eta y$ and therefore with J, $\equiv_J$ is a congruence relation also with respect to the Stonean negation operator.

¿From this result we can directly infer:

Corollary 3.1. Let $\mathcal{A}$ be a Stonean MV-algebra and J a prime s-ideal belonging to $\mathcal{A}$, the natural homomorphism from $\mathcal{A}$ onto the quotient linear MV-algebra (then Stonean) $\mathcal{A}/J$ preserves the Stonean negation operator.

In an equivalent way to the traditional case presented by C.C.Chang in [Ch58] we shall build for any Stonean MV-algebra an isomorphic subdirect product of quotient MV-algebras with prime s-ideals.

We provide a technique to generate s-ideals:

Definition 3.2. Let $\mathcal{A}=< A, \oplus, \neg, \mathbf{0} >$ a Stonean MV-algebra, let J an ideal of $\mathcal{A}$, we define $J^0 = \{x \in A \mid \exists y \in J : x \leq \eta\eta y\}$.

By (P6) we have immediately that $J \subseteq J^0$.

Theorem 3.2. Let J an ideal of a Stonean MV-algebra $\mathcal{A}=< A, \oplus, \neg, \mathbf{0} >$. J^0 is the smallest s-ideal containing J.

Proof:
First, we prove that J^0 is an ideal.
Suppose $a, b \in J^0$. Thus $\exists x, y \in J$ s.t. $a \leq \eta\eta x$ and $b \leq \eta\eta y$. Then by monotony of $\oplus$ (Theorem 1.8 in [Ch58]) and respectively (P5), (B1) and (P3) $a \oplus b \leq \eta\eta x \oplus \eta\eta y = \eta(\eta x \odot \eta y) = \eta(\eta x \wedge \eta y) = \eta\eta(x \vee y)$. Clearly $x \vee y \in J$ since $x, y \in J$. Hence $a \oplus b \in J^0$ and (I3) holds. Trivially (I1) and (I2) hold too. Thus J^0 is an ideal.
Now we prove that J^0 is an s-ideal.
Suppose $a \odot b \in J^0$. Then $\exists x \in J$ s.t. $a \odot b \leq \eta\eta x$. Since (P6) $\eta\neg b \leq b$, we have $a \odot \eta\neg b \leq \eta\eta x$. Now by (B1) $a \odot \eta\neg b = a \wedge \eta\neg b$. Thus $\eta\eta(a \odot \eta\neg b) = \eta\eta(a \wedge \eta\neg b) \leq \eta\eta x$. Now if we join (P2) and (P3) we have $(a \wedge \eta\neg b) = \eta\eta a \wedge \eta\neg b$. Thus $\eta\eta a \odot \eta\neg b \leq \eta\eta a \wedge \eta\neg b \leq \eta\eta x$ and $\eta\eta a \odot \eta\neg b \in J^0$.

It remains to show that J^0 is the smallest s-ideal containing J.

Clearly $J \subseteq J^0$. Let I be an s-ideal of $\mathcal{A}$, s.t. $J \subseteq I$. We want to show that $J^0 \subseteq I$. Let $a \in J^0$. Then $a \leq \eta\eta x$ for some $x \in J$. Then $x \in I$. Since I is an s-ideal and $x = x \odot \mathbf{1}$, $\eta\eta x \in I$. Thus $a \in I$.

To prove the next theorem we need to show the following results:

Lemma 3.2. In any MV-algebra $\mathcal{A}$, $\forall a, x, y, z \in A$,
$a \leq x \oplus y, a \leq x \oplus z \Rightarrow a \leq x \oplus (y \wedge z)$.

Proof:
By monotony of $\wedge$ (Theorem 1.5 of [Ch58]) and axiom 11 of [Ch58] we have $a = a \wedge a \leq (x \oplus y) \wedge (x \oplus z) = x \oplus (y \wedge z)$.

Lemma 3.3. In any MV-algebra $\mathcal{A}$, $\forall x, y, \in A$, $\forall m \in N$,
$m(\neg x \odot y) \wedge m(x \odot \neg y) = \mathbf{0}$.

Proof:
For duality by Theorem 3.7 of [Ch58].

Lemma 3.4. In any Stonean MV-algebra $\mathcal{A}$, $\forall x, y, \in A$,
$x \wedge y = \mathbf{0} \Rightarrow \eta\eta x \wedge \eta\eta y = \mathbf{0}$.

Proof:
If $x \wedge y = \mathbf{0}$ then $\eta\eta(x \wedge y) = \mathbf{0}$. By (P2) we have $\eta(\eta x \vee \eta y) = \mathbf{0}$. Then, by (P3), $\eta\eta x \wedge \eta\eta y = \mathbf{0}$.

Theorem 3.3. Let $\mathcal{A}$ be a Stonean MV-algebra. $\forall a \in A$, $a \neq \mathbf{0}$, there is a prime s-ideal $J \subset A$ s.t. $a \notin J$.

Proof:
Suppose $a \neq \mathbf{0}$. Now by Lemma 3.1 $\{\mathbf{0}\}$ is an s-ideal. By Zorn's Lemma, there is an s-ideal J which is maximal w.r.t. the property "$a \notin J$".

Suppose, by ctr., $\exists x, y \in A$ s.t. $\neg x \odot y \notin J$ and $x \odot \neg y \notin J$.

We define $J^{\sim}_{\neg x \odot y}$:= $(Id(J \cup \{\neg x \odot y\}))^0$ and $J^{\sim}_{x \odot \neg y}$:= $(Id(J \cup \{x \odot \neg y\}))^0$.

By Theorem 3.2, $J^{\sim}_{\neg x \odot y}$ and $J^{\sim}_{x \odot \neg y}$ are s-ideals containing $Id(J \cup \{\neg x \odot y\})$ and $Id(J \cup \{x \odot \neg y\})$.

By definition of maximality of J, $a \in J^{\sim}_{\neg x \odot y}$ and $a \in J^{\sim}_{x \odot \neg y}$.

Thus, $\exists r \in Id(J \cup \{\neg x \odot y\})$ and $\exists s \in Id(J \cup \{x \odot \neg y\})$ s.t. $a \leq \eta\eta r$ and $a \leq \eta\eta s$.

Now $r = i \oplus n(\neg x \odot y)$ for some $i \in J$ and $n \in N$, $s = j \oplus m(x \odot \neg y)$ for some $j \in J$ and $m \in N$.

It implies, by (P5), that $a \leq \eta\eta(i \oplus n(\neg x \odot y)) = \eta\eta i \oplus \eta\eta(n(\neg x \odot y))$ and $a \leq \eta\eta(j \oplus m(x \odot \neg y)) = \eta\eta j \oplus \eta\eta(m(x \odot \neg y))$.

Let $k = max\{n, m\}$, then $a \leq (\eta\eta i \oplus \eta\eta j) \oplus \eta\eta(k(\neg x \odot y))$ and $a \leq (\eta\eta i \oplus \eta\eta j) \oplus \eta\eta(k(x \odot \neg y))$.

By Lemma 3.2 we have $a \leq (\eta\eta i \oplus \eta\eta j) \oplus (\eta\eta(k(\neg x \odot y)) \wedge \eta\eta(k(x \odot \neg y)))$.

By Lemma 3.3 $k(\neg x \odot y) \wedge k(x \odot \neg y) = \mathbf{0}$.

By Lemma 3.4 $\eta\eta(k(\neg x \odot y)) \wedge \eta\eta(k(x \odot \neg y)) = \mathbf{0}$. Hence $a \leq \eta\eta i \oplus \eta\eta j$.

Since J is a s-ideal and $i, j \in J$, $\eta\eta i \in J$ and $\eta\eta j \in J$. Thus $\eta\eta i \oplus \eta\eta j \in J$. By (I2) we have $a \in J$, against our ab absurdo hypothesis. Then J is a prime s-ideal.

Representation Theorem for Stonean MV-algebras. Every Stonean MV-algebra $\mathcal{A}$ is isomorhic to a subdirect product of Stonean linear MV-algebras s.t. $\forall x \in A$, $x =< x_1, ..., x_n, .. > \Rightarrow \eta < x_1, ..., x_n, .. > = < \eta_0 x_1, ..., \eta_0 x_n, .. >$.

Proof:

A particular case of a theorem of Universal Algebra, due to Birkhoff [Bi67], is shown in [CDM00] (Theorem 1.3.2); it tells us that a MV-algebra $\mathcal{A}$ is isomorphic to a subdirect product of a family of linear MV-algebras if there exists a family of prime ideals $\{J_i \mid i \in N\}$ of $\mathcal{A}$ s.t. $\bigcap J_i = \{\mathbf{0}\}$. By the previous Theorem, for any $x \in A$ there is a prime s-ideal J s.t. $x \notin J$. Then $\{\mathbf{0}\}$ is the intersection of all the prime s-ideals of $\mathcal{A}$ maximal with respect to the property "$x \notin J$". Hence $\mathcal{A}$ is isomorphic to a subdirect product of quotient MV-algebras built with its prime s-ideals. These MV-algebras are the homomorphic images of $\mathcal{A}$, related to the natural homomorphismes. These homomorphismes, by Corollary 3.1, preserve the Stonean negation operator. Thus our proof is complete.

4. Algebraic Completeness

We will prove that an equation defined on our enriched language with Stonean operator holds in any MV-algebra if it holds in the standard MV-algebra. We will follow the track of Chang's completeness Theorem [Ch59]. Then we assume familiarity with this proof and with all the results utilized to pursue it (see also [CDM94]). We recall that Chang's proof is not self-contained but he exploits the completeness of the first order theory of divisible totally ordered Abelian groups (Chang's references are [Ta31] and [Ta56] but, as reported in footnote at page 79 [Ch59], Tarski's proof has never appeared explicitly, then for a clear presentation of this result we advise the readers to consult appendix at page 91 of [CDM94]). As fundamental step of his proof, C.C.Chang had build a totally ordered Abelian group made of infinite copies of an MV-algebra. We introduce this expedient:

Definition 4.1. Let $\mathcal{A}$ be a linear MV-algebra. The algebraic structure $\mathcal{G}_A$ is defined in the following way:

$G_A \stackrel{\text{def}}{=} \{(n,x) \mid n \in Z, x \in \mathrm{A} - \{\mathbf{1}\}\}$

Its operators are defined as:

$$(m,x)+(n,y) \stackrel{\text{def}}{=} \begin{cases} (n+m, x \oplus y) & \text{if } x \oplus y \neq \mathbf{1} \\ (n+m+1, x \odot y) & \text{if } x \oplus y = \mathbf{1} \end{cases}$$

$$-(n,x) \stackrel{\text{def}}{=} \begin{cases} (-n, \mathbf{0}) & \text{if } x = \mathbf{0} \\ (-(n+1), \neg x) & \text{if } \mathbf{0} \neq x \neq \mathbf{1} \end{cases}$$

and its related order relation is:

$(n,x) \sqsubseteq (m,y) \quad \stackrel{\text{def}}{\Leftrightarrow} \quad n < m \text{ or, } n = m \text{ and } x \leq y$

C.C.Chang in [Ch59] proved that $\mathcal{G}_A$=$< G_A, +, -, \sqsubseteq, (0,\mathbf{0}) >$ is a totally ordered Abelian group;

Moreover if we define:

Definition 4.2. Let $\mathcal{G}$=$< G, +, -, 0, \sqsubseteq >$ be a totally ordered Abelian group, $u \in G$:

$\Gamma(\mathrm{G}, u) \stackrel{\text{def}}{=} \{x \in G \mid 0 \sqsubseteq x \sqsubseteq u\}$

$\neg x \stackrel{\text{def}}{=} u - x$

$x \oplus y \stackrel{\text{def}}{=} min\{u, x+y\}$

we can immediately verify that $< \Gamma(\mathrm{G}, u), \oplus, \neg, 0 >$ is a linear MV-algebra.

$u \in G$ is a *strong unit* iff for any $x \in G$ there exists an $n \in N$ s.t. $x \sqsubseteq nu$.

$\mathcal{G}_A$ is composed of infinite copies of $\mathcal{A}$; $\Gamma(\mathrm{G}_A, (1,\mathbf{0}))$ belongs to them, then we have:

Theorem 4.1. If $\mathcal{A}$ is a linear MV-algebra, $\Gamma(\mathrm{G}_A, (1,\mathbf{0}))$ is isomorphic to $\mathcal{A}$.

This result can be generalized to:

Theorem 4.2. If u is the strong unit of $\mathcal{G}$, there exists an isomorphism f from $\mathcal{G}$ onto $\mathcal{H} = \mathcal{G}_{\Gamma,(\mathrm{G},u)}$:
i) $f(u) = (1, \mathbf{0})$
ii) $x \sqsubseteq y$ in $\mathcal{G} \Leftrightarrow f(x) \sqsubseteq f(y)$ in $\mathcal{H}$

Proof:
(See either Theorem 2.4.10 in [CDM94] or [Ch59]).

The first order language of totally ordered Abelian groups theory L' is composed by the usual logic symbols and 0,+,-,$\sqcap$, $\sqcup$ with their traditional meaning. We have to fix their corresponding definitions:

Definition 4.3. A language L of a Stonean MV-algebra $\mathcal{A}$ is composed by:
$\mathbf{0}$: costant
$x_1, ..., x_n, ..$: variables
$\neg$: unary functor
$\oplus$: binary functor
η : unary functor.

We define inductively an *MV$_S$-term*:
1) $\mathbf{0}$, $x_1, ..., x_n, ..$ are MV$_S$-terms.
2) If x_i is an MV$_S$-term, then $\neg x_i$ is an MV$_S$-term.
3) If x_i and x_j are MV$_S$-terms, then $x_i \oplus x_j$ is an MV$_S$-term.
4) If x_i is an MV$_S$-term, then ηx_i is an MV$_S$-term.

Let p be an MV$_S$-term containing the variables $x_1, ..., x_t$ and assume $a_1, ..., a_t$ are elements of $\mathcal{A}$. Substituting an element $a_i \in A$ for all occurrences of the variable x_i in p, for $i = 1, ..., t$, by the above rules 1)-4) and interpreting the symbols $\mathbf{0}$,$\oplus$, $\neg$ and η as the corresponding operations in $\mathcal{A}$, we obtain an element of A, denoted $p^{\mathcal{A}}(a_1, ..., a_t)$. In more detail, proceeding by induction on the number of operation symbols occurring in p, we define $p^{\mathcal{A}}(a_1, ..., a_t)$ as follows:
i) $x_i^{\mathcal{A}} = a_i$, for each $i = 1, ..., t$;
ii) $(\neg p)^{\mathcal{A}} = \neg(p^{\mathcal{A}})$;
iii) $(p \oplus q)^{\mathcal{A}} = (p^{\mathcal{A}} \oplus q^{\mathcal{A}})$;
iv) $(\eta p)^{\mathcal{A}} = \eta(p^{\mathcal{A}})$.
By the above definition, given any Stonean MV-algebra $\mathcal{A}$ we can associate each MV$_S$-term in the variables $x_1, ..., x_n$ with a function $p^{\mathcal{A}} : A^n \mapsto A$. These functions are called *term functions on* A.

An *MV$_S$-equation on variables* $x_1, ..., x_t$ is an expression $p = q$, where p and q are MV$_S$-term containing at most the variables $x_1, ..., x_t$.

We say that an MV-algebra $\mathcal{A}$ *satisfies* an MV$_S$-equation $p = q$ (we write $\mathcal{A}\models p = q$) if and only if for any sequence of elements $(a_1, ..., a_t) \in A$, $p^{\mathcal{A}}(a_1, ..., a_t) = q^{\mathcal{A}}(a_1, ..., a_t)$.

Theorem 4.3. If a Stonean MV-algebra $\mathcal{A}$ is subdirect product of a family of linear MV-algebras (Stonean) $\{\mathcal{A}_i \mid i \in \mathrm{I}\}$, then $\mathcal{A}\models p = q \Leftrightarrow$ for any i $\mathcal{A}_i \models p = q$.

Proof:

In Chang's subdirect representation Theorem there is a homomorphism from $\mathcal{A}$ onto any linear MV-algebra of its subdirect product. Since we have proved that the Stonean negation operator is preserved too into these structures, every MV$_S$-equation continues to hold in any $\mathcal{A}_i$.

Vice versa if an MV$_S$-equation holds in any $\mathcal{A}_i$, it holds in their direct product $\Pi_{i \in I}\mathcal{A}_i$. Since $\mathcal{A}$ is isomorphic to a subalgebra of $\Pi_{i \in I}\mathcal{A}_i$, it holds in $\mathcal{A}$.

Corollary 4.1. An MV$_S$-equation is satisfied in any Stonean MV-algebras if and only if it is satisfied in any linear (Stonean with η_0) MV-algebra.

We will report in the following steps Chang's completeness Theorem, as it has been presented in [CDM94], to check its validity with respect to the Stonean extension. Every totally ordered Abelian group can be embedded into a divisible totally ordered Abelian group. From the completeness of the first order theory of these last structures follows that every universal sentence of the first order theory of totally ordered Abelian groups is satisfied in the additive group Q of rational numbers if and only if it is satisfied in any totally ordered Abelian group [Ch59]. Then any MV$_S$-equation has to be associated to an universal sentence of the first order language of totally ordered Abelian groups theory (we will call it L') to exploit its completeness. We will do it by induction on the degree of complexity of an MV$_S$-term.

Definition 4.4. The *degree of complexity* of an MV$_S$-term p: $d(p) \stackrel{\text{def}}{=}$ the number of times that symbols $\oplus$, $\neg$ and η appear in p.

We associate to any MV$_S$-term p a term $p' \in L'$ by induction on the degree of complexity of p:

If $d(p)$=0 (p=0 or $p = x_i$) then $p' = p$.

We suppose to have associated MV$_S$-terms until degree of complexity n:

if $d(p)$=$n + 1$, we can have either:

1) $p = \neg q$ with $d(q)$=n or

2) $p = q \oplus r$ with $d(q) \leq n$ and $d(r) \leq n$ or

3) $p = \eta q$ with $d(q)$=n

Let z be a free variable that belongs to L', we define

in case 1): $p' = z - q'$;
in case 2): $p' = z \sqcap (q' + r')$;
in case 3): $p' = \begin{cases} z & \text{if } q' = 0 \\ 0 & \text{otherwise} \end{cases}$

Then we define $\alpha_{pq} \stackrel{\text{def}}{=} \forall x_1, ..., x_n (0 \sqsubseteq x_i \sqsubseteq z \bigwedge, ..., \bigwedge 0 \sqsubseteq x_n \sqsubseteq z) \rightarrow p' = q'$.

It can be easily checked, by the way $\mathcal{G}_A$ has been built, that the following sentence holds:

Proposition 1 Let $\mathcal{A}$ be a Stonean MV-algebra, let $p = q$ an MV$_S$-equation;
$\mathcal{A} \models p = q \quad \Leftrightarrow \quad \alpha_{pq}(z)$ is true in $\mathcal{G}_A$ when we attribute to z the value $(1, \mathbf{0})$.

At last we can introduce:

Completeness Theorem. An MV$_S$-equation is satisfied in any Stonean MV-algebra if and only if it is satisfied in the standard MV-algebra enriched with the Stonean negation η_0.

Proof:
$\Leftarrow$ (not trivial) : By contradiction we suppose there is an MV-algebra $\mathcal{A}$ such that $\mathcal{A} \not\models p = q$. From Corollary 4.1 we infer that there is a linear MV-algebra $\mathcal{B}$ s.t. $\mathcal{B} \not\models p = q$.

By Proposition 1 above there is an universal sentence β of the 1^o order theory of the totally ordered Abelian groups, $\beta = \forall z > 0 \ \alpha_{pq}(z)$ s.t. β is false in $\mathcal{G}_B$, and hence, by the completeness of totally ordered Abelian groups, β is false in Q (group of rational numbers with usual operations).

It means that there is a $c > 0, c \in Q$ s.t. c does not verify β in Q. Let's consider f: $Q \mapsto Q$ defined by $f(x) \stackrel{\text{def}}{=} c^{-1}x$. $f(c) = 1$. f is an isomorphism from Q onto itself (antiautomorphism), then f preserves falsity of sentences and therefore β is false in Q when we attribute to z the value $1 \in Q$. By Theorem 4.2 Q is isomorphic to $\mathcal{G}_{\Gamma(Q,1)}$. Thus β is false in $\mathcal{G}_{\Gamma(Q,1)}$ with $z = 1$ and, by Proposition 4.1 above, $\Gamma(Q, 1) = \mathcal{A}_{[0,1]} \not\models p = q$.

REFERENCES

[Ba96] M.Baaz, *Infinite-valued Gödel logics with 0-1-projections and relativizations.*, In GÖDEL '96 - Logical Foundations of Mathematics, Computer Science and Phisycs; Lecrure Notes in Logic **6** (1996), P.Hajek, Ed.Springer, pp.23-33.

[Be97] L.P.Belluce, *Generalized fuzzy connectives on MV-algebras* , Journal of Mathematical Analysis and Applications **206:1** (Feb 1 1997), pp.485-499.

[Bi67] G.Birkhoff, "Lattice theory". 3.ed.Providence: Am.Math.Soc.1967.

[CDM94] R.Cignoli,I.D'Ottaviano,D.Mundici "Álgebras das Lógicas de Łukasiewicz". Colecao CLE 12, Campinas (Brazil) 1994.

[CDM00] R.Cignoli,I.D'Ottaviano,D.Mundici "Algebraic Foundation of Many-valued Reasoning". Trends in logic-Studia logica library. Kluwer Academic Publishers, Dodrecht/Boston/London 2000.

[CGP98] G.Cattaneo,R.Giuntini,R.Pilla, *BZMV*^{dM} *algebras and Stonean MV-algebras (Applications to fuzzy sets and rough approximations)*, Fuzzy Sets and Systems **108**, 1998, pp.201-222.

[Ch58] C.C.Chang, *Algebraic Analysis of many valued logics*, Transactions of the American Mathematical Society **88** (1957), pp.467-490.

[Ch59] C.C.Chang, *A new proof of the completeness of Łukasiewicz axioms*, Transactions of the American Mathematical Society **93** (1958), pp.74-80.

[DH01] J.M.Dunn, G.M.Hardegree, "Algebraic Methods in Philosophical Logic". OXFORD LOGIC GUIDES 41 - CLARENDON PRESS - OXFORD 2001.

[EGH96] F.Esteva, L.Godo and P.Hajek *A complete many-valued fuzzy logic with product conjunction*, Archive for Mathematical Logic **35** (1996), pp.1-19.

[EGHN00] F.Esteva, L.Godo, P.Hajek and M.Navara *Residuated fuzzy logics with an involutive negation*, Arch. for Math. Logic **39** (2000), pp.103-124.

[EGM01] F.Esteva, L.Godo, F.Montagna, *The* $L\Pi$ *and* $L\Pi\frac{1}{2}$ *logics: two complete fuzzy systems joining Łukasiewicz and Product Logics*, Archive for Mathematical Logic **40** (2001), Ed. Springer-Verlag, pp.39-67.

[Go33] K.Gödel, *Zum intuitionistischen Aussagenkalkül*, Anzeiger Akademie der Wissenschaften Wien, Math.-Naturwissensch, Klasse **69** (1933), pp.65-69.

[Ha98] P.Hajek, "Metamathematics of Fuzzy Logic". Trends in logic-Studia logica library. Kluwer Academic Publishers, Dodrecht/Boston/London, 1998.

[He82] I.N.Herstein "Algebra". Editori Riuniti, Roma 1982.

[Ma73] P.Mangani, *Su certe algebre connesse con logiche a più valori*, Bollettino dell'Unione Matematica Italiana **8** (1973), pp.68-78.

[Ov83] S.V.Ovchinnikov, *General negation in fuzzy set theory* , J.Math.Anal.and appl. **92** (1983), pp.234-239.

[Pa79] J.Pavelka, *On Fuzzy Logics: I, II, III* , Zeitr. f. Math. Log. und Grundlagen der Math. **25** (1979), pp.45-52, pp.119-134, pp.447-464.

[TT92] G.Takeuti and S.Titani, *Fuzzy Logic and Fuzzy Set Theory*, Archive for Mathematical Logic **32** (1992), pp.1-32.

[Ta31] A.Tarski, *Sur les ensembles définissable de nombre réels* I, Fund. Math. vol.17 (1931), pp.210-239.

[Ta56] A.Tarski, “Logics, semantics, metamathematics”. Oxford: Clarendon Press 1956.