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Geometric non commutative phase spaces

Abstract The aim of this paper is to describe some geometric examples of non commutative and cyclic phase spaces, filling a gap in the literature and developing the project of geometrization of semantics for linear logics started in [12]. Besides, we present an algebraic semantics for non commutative linear logic with exponentials.

Keywords: geometric models and examples, semantical methods, phase spaces, linear logic, affine linear logic, non commutative linear logic.

§0. INTRODUCTION

The aim of this paper is to describe some geometric examples of non commutative and cyclic phase spaces, filling a gap in the literature and developing the project of geometrization of semantics for linear logics started in [12]. Besides, we present an algebraic semantics for non commutative linear logic with exponentials (**NLL**), that is the logic obtained from a sequent formulation of classical logic by rejecting and *recovering* all structural rules (weakening, contration and exchange). Moreover we sketch how this semantics is uniform over related logics with exponentials, such as: i) cyclic linear logic with (**CyLL**), a variant of non commutative linear logic in which a restricted form of exchange, cyclic exchange, is allowed; ii) linear logic (**LL**), which rejects and recovers weakening and contraction rules; iii) affine linear logic (**AL**), which lacks contraction rules. Since in all these logics the exponentials play the crucial role of recovering and controlling the logical power of the dismissed structural rules, we avoid to use the misleading term "substructural" for this class of logics.

It is well-known that an algebraic semantics for a logical calculus ensures that the notion of truth, which is *infinite* and *static* in character, and the notion of provability, which is *finite* and *dynamic*, correspond to each other. In order to build a "bridge" between these two notions, an infinite class of algebraic structures (models) is introduced, which is – hopefully – as "culturally distant" from its syntactical counterpart as possible. By virtue of this correspondence, using the machinery of algebra, we can state syntactical properties relative to provability; and, vice versa, provable logical principles lead to algebraic properties, i.e. properties of the models. In particular, in order to prove that a formula A is unprovable, it suffices to exhibit a model in which A is not valid. If our semantics is based on mathematical structures that are simple and traditionally known, then the search of suitable models becomes easier. The semantics we present is a development of the phase semantics introduced by Girard for linear logic [4, 5]. The crucial difference lies in the fact that a Girardian phase space consists simply of a monoid M and a subset $\perp$ of M (without provisos on M and $\perp$), whereas now it is imposed a preorder on M. In particular, one requires that the monoid operation • be monotone and that $\forall x, y \in M$:

i) if $x \leq y$ and $y \in \perp$, then $x \in \perp$;
ii) $\sigma(x \bullet y) = \sigma(x) \bullet \sigma(y)$, $\sigma(u)=u$, where $\sigma : M \longrightarrow M$ is a bijection;
iii) $x \leq y$ iff $\sigma(x) \leq \sigma(y)$.

Then, by introducing a special subset of M – the set **K** = $\{x\colon \forall y \in M\ x \bullet y \leq y \bullet x$ and $y \bullet x \leq x \bullet y,\ x \leq u,\ x \leq x \bullet x\}$ – it is possible to define in a natural way the exponentials. The motivation for enriching Girard's phase space with preorders comes from the goal to find a semantical framework which is common to different, but correlated logics. For a general introduction to the substructural landscape see [11].

§1. PHASE SPACES AND PHASE SPACES WITH PREORDERS

1.1. PRELIMINARIES: NON COMMUTATIVE PHASE SPACES

(Commutative) phase spaces are introduced by J.Y. Girard in order to modelize **LL** [4, 5]. Non commutative and cyclic phase spaces play an analogous role with respect to **NLL** [1,14] and **CyLL** [16]. Genuine (mathematical) examples of non commutative and cyclic phase spaces are given in subsection 1.2.

Definition 1.1. By *non commutative phase space*, we mean every structure $(M, \bullet, u, \sigma, \perp)$ where:

(i) $(M, \bullet, u)$ is a monoid;

(ii) $\sigma : M \longrightarrow M$ is a bijection;

(iii) $\perp \subseteq M$;

(iv) $\forall x, y \in M$: $x \bullet y \in \perp$ if and only if $y \bullet \sigma(x) \in \perp$ (σ-cyclicity).

A non commutative phase space $(M, \bullet, u, \sigma, \perp)$ will be simply indicated with $(M, \sigma, \perp)$.

Examples 1.1.1. Every structure $(G, (\_)^{-1}, H)$, where G is a monoid reduct of a group and H a subset of G s.t.

$$(*)\quad \forall y \in G\colon y \bullet H \bullet y \subseteq H,$$

is a *non commutative phase space*. In fact: if $x \bullet y \in H$, then $x^{-1} \bullet x \bullet y \bullet x^{-1} = y \bullet x^{-1} \in H$. Vice versa, if $y \bullet x^{-1} \in H$, then $x \bullet y \bullet x^{-1} \bullet x = x \bullet y \in H$. Given a group G, a subset H with the property (*) always exists. To show that, we posit for every $n \geq 1$:

$H_n = \{x_1 \bullet x_2 \bullet \ldots \bullet x_{n-1} \bullet x_n \bullet x_n \bullet x_{n-1} \bullet \ldots \bullet x_2 \bullet x_1 : x_1, x_2, \ldots, x_{n-1}, x_n \in G\}$

(For instance, H_2 is the subset of the squares). The subset H defined in this way: $H = \bigcup_{n \in M} M_n$ is a *non commutative phase space*.

Definition 1.1.2 A *non commutative phase space* $(M, \sigma, \perp)$ is called *cyclic phase space* if $\sigma = id_M$. That is, $\forall x, y \in M$, $x \bullet y \in \perp$ if and only if $y \bullet x \in \perp$ (*cyclicity*).

A cyclic phase space will be simply indicated with $(M, \perp)$.

Examples 1.1.3. I) Let G be a group with an element x_0 s. t. $\forall y \in G$: $x_0 \bullet y = y \bullet x_0$. Then, the structure $(G, \{x_0\})$ is a cyclic phase space. II) More generally, every structure (G, H), where G is a group and H is any of its normal subgroups, is a cyclic phase space.

1.2. A GEOMETRICAL CONSTRUCTION

I). Let **A** denote the finite sequence of intervals of the real line:

$$[a_1,b_1), [a_2,b_2), \ldots, [a_n,b_n)$$

where $d_i = b_i - a_i > 0, \quad \forall i = 1, \ldots, n$.

We consider the function $\gamma_{\mathbf{A}} : [0, d_1+d_2+\ldots+d_n) \longrightarrow \mathbb{R}$ defined in the following way:

$$\gamma_{\mathbf{A}}(t) = \begin{cases} t+a_1 & 0 \leq t \leq d_1, t + d_2 - d_1 \qquad d_1 \leq t \leq d_1+d_2 \\ \quad\vdots & \quad\vdots \\ t + d_n - (d_1+d_2+\ldots+d_n) & d_1 + d_2 + \ldots + d_{n-1} \leq t \leq d_1+d_2+\ldots+d_n \end{cases}$$

Let h be the function: $h : \mathbb{R} \longrightarrow \mathbb{R}^3$

$$u \longmapsto (\cos(u), \sin(u), u)$$

The function h describes a *cylindrical helix* of radius 1 and axis coinciding with the axis *z*.

Then, we can associate with every finite sequence $\mathbf{A} \equiv [a_1,b_1), [a_2,b_2),..., [a_n,b_n)$ the function:

$$h_{\mathbf{A}} : [0, d_1+d_2+...+d_n) \longrightarrow \mathrm{IR}^3$$

where $h_{\mathbf{A}} \equiv h \circ \gamma_{\mathbf{A}}$. Such a function describes a finite sequence of tracts of the circular helix. Now, let M = $\{h_{\mathbf{A}}$: **A** finite sequence of intervals$\}$. Notice that the empty function h_e associated with the empty sequence *e* is contained in M.

In M, a binary operation • may be trivially defined as follows: $\forall \mathbf{A}, \mathbf{B}$, $h_{\mathbf{A}} \bullet h_{\mathbf{B}} \equiv h_{\mathbf{AB}}$ where **AB** is the concatenation of **A** and **B**. It easy provable that if $h_{\mathbf{A}} \equiv h_{\mathbf{A'}}$ and $h_{\mathbf{B}} \equiv h_{\mathbf{B'}}$, then $h_{\mathbf{AB}} \equiv h_{\mathbf{A'B'}}$. Therefore it follows that the structure (M, •, h_e) is a (non commutative) monoid.

Let us consider the function $\sigma : M \longrightarrow M$, given by $\sigma(h_{\mathbf{A}}) = \sigma(h_{\mathbf{A}-2\pi})$. It is easy to show that σ is bijective.

Finally, we posit $\perp = \{h_{\mathbf{A}}: h_{\mathbf{A}}$ is continuous and $pr_{x,y}(\mathrm{Im}(h_{\mathbf{A}})) = \mathbf{C}\}$, where **C** is the circle of the plane xy of center (0,0,0) and radius 1. It is easy to observe that the structure (M, σ, $\perp$) is a non commutative phase space.

Remark 1.2.1. In such a structure the following property does *not* hold:

$$\forall h_{\mathbf{A}}, h_{\mathbf{B}} \in M, \quad h_{\mathbf{A}} \bullet h_{\mathbf{B}} \in \perp \text{ implies } h_{\mathbf{B}} \bullet h_{\mathbf{A}} \in \perp.$$

In fact, in general, when $h_{\mathbf{A}} \bullet h_{\mathbf{B}}$ is a continuous function, $h_{\mathbf{B}} \bullet h_{\mathbf{A}}$ need not to be continuous.

II). As before, let **A** denote the finite sequence of intervals of the real line:

$$[a_1,b_1), [a_2,b_2),..., [a_n,b_n)$$

where $\forall$ i =1,...,n $d_i = b_i - a_i > 0$. We consider the same function defined above:

$$\gamma_{\mathbf{A}}: [0, d_1+d_2+...+d_n) \longrightarrow \mathrm{IR}$$

Let h be now the function: $h: \mathrm{IR} \longrightarrow \mathrm{IR}^2$

$$u \longmapsto (\cos(u), \sin(u))$$

The function h describes the (oriented) circle **C** of radius 1 and center n in the origin. We can associate with any finite sequence $\mathbf{A} \equiv [a_1,b_1), [a_2,b_2),..., [a_n,b_n)$ the function $h_{\mathbf{A}} \equiv h \circ \gamma_{\mathbf{A}}$.

This function describes a finite sequence of oriented arcs of the circle **C**.

Let M = {$h_{\mathbf{A}}$: **A** is a finite sequence of intervals}. We define in M the binary operation •: $\forall$**A**,**B**, $h_{\mathbf{A}} \bullet h_{\mathbf{B}} \equiv h_{\mathbf{AB}}$, where **AB** is the concatenation of **A** and **B**. It can be easily shown that if $h_{\mathbf{A}} \equiv h_{\mathbf{A'}}$ and $h_{\mathbf{B}} \equiv h_{\mathbf{B'}}$, then $h_{\mathbf{AB}} \equiv h_{\mathbf{A'B'}}$. Hence, it follows that the structure (M, •, h_e) is a (non commutative) monoid. Finally, we posit $\perp$ = {$h_{\mathbf{A}}$: $h_{\mathbf{A}}$ is continuous and $\mathrm{Im}(h_{\mathbf{A}})$ = **C**}.

It is easy to see that the structure (M, $\perp$) is a cyclic phase space.

1.3. <u>OPERATIONS ON THE POWER SET OF THE MONOID AND FACTS</u>

Definition 1.3.1. Given a non commutative phase space (M, σ, $\perp$), we define the following operations on the power set *P*(M). Given A, B$\subseteq$M:

$A \bullet B = \{a \bullet b: a \in A \text{ and } b \in B\}$;

$A \to_r B = \{x: \forall a \in A \ \ a \bullet x \in B\}$;

$A \to_l B = \{x: \forall a \in A \ \ x \bullet a \in B\}$;

$A^{\perp} = A \to_r \perp = \{x: \forall a \in A \ \ a \bullet x \in \perp\}$;

${}^{\perp}A = A \to_l \perp = \{x: \forall a \in A \ \ x \bullet a \in \perp\}$.

By *fact,* we mean every A$\subseteq$M such that $A = {}^{\perp}A^{\perp}$. It easy to show that A$\subseteq$M is a fact iff $\exists$ B$\subseteq$M such that $A = B^{\perp}$ or $A = {}^{\perp}B$. The following subsets of M: $\perp$; $1 = {}^{\perp}\{u\}^{\perp}$; $0 = {}^{\perp}\varnothing^{\perp}$; T = M are all facts.

Proposition 1.3.2. In every non commutative phase space (M, σ, $\perp$), for any A, B$\subseteq$M:

(i) if $A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$ and ${}^{\perp}B \subseteq {}^{\perp}A$;

(ii) ${}^{\perp}(A^{\perp}) = ({}^{\perp}A)^{\perp}$.

Proof: left as exercise.

Proposition 1.3.3. In every non commutative phase space (M, σ, $\perp$) the function:

$${}^{\perp}(\_)^{\perp}: P(M) \longrightarrow P(M)$$

satisfies the following properties:

(i) for any $A \subseteq M$: $A \subseteq {}^{\perp}A^{\perp}$;

(ii) for any A, $B \subseteq M$: if $A \subseteq B$ then ${}^{\perp}A^{\perp} \subseteq {}^{\perp}B^{\perp}$;

(iii) for any $A \subseteq M$: ${}^{\perp\perp}A^{\perp\perp} \subseteq {}^{\perp}A^{\perp}$;

Proof: left as exercise.

Definition 1.3.4. Given a non commutative phase space (M, σ, $\perp$) for every pair of facts A, B, we posit:

(i) $A \otimes B = {}^{\perp}(A \otimes B)^{\perp}$;

(ii) $A \wp B = ({}^{\perp}B \otimes {}^{\perp}A)^{\perp} = {}^{\perp}(B^{\perp} \otimes A^{\perp})$;

(iii) $A \oplus B = {}^{\perp}(A \oplus B)^{\perp}$;

(iv) $A \& B = A \cap B$.

Proposition 1.3.5. The set of facts is closed with respect to the operations: $\to_r$, $\to_l$, $(-)^{\perp}$, ${}^{\perp}(-)$, $\otimes$, $\oplus$, , $\&$, $\wp$. Moreover, $A \to_r B = A^{\perp} \wp B$ and $A \to_l B = B \wp {}^{\perp}A$. It easy to check that De Morgan Laws hold. Moreover, for every fact A, B, C and D:

(i) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$;

(ii) $A \otimes 1 = A = 1 \otimes A$;

(iii) $A \subseteq B$ and $C \subseteq D$ implies $A \otimes C \subseteq B \otimes D$;

(iv) $(A \& B) \& C = A \& (B \& C)$;

(v) $A \& T = A = T \& A$;

(vi) $A \& B \subseteq A$ and $A \& B \subseteq B$;

(vii) $C \subseteq A$ and $C \subseteq B$ implies $C \subseteq A \& B$;

(viii) $A \subseteq B$ and $C \subseteq D$ implies $A \& C \subseteq B \& D$;

(ix) $A \& B = B \& A$;

(x) $A \otimes 0 = 0 = 0 \otimes A$;

(xi) $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$;

(xii) $(B \oplus C) \otimes A = (B \otimes A) \oplus (C \otimes A)$;

(xiii) $A \otimes (B \wp C) \subseteq (A \otimes B) \wp C$;

(xiv) $A \otimes (B \& C) \subseteq (A \otimes B) \& (A \otimes C)$;

(xv) $(B \& C) \otimes A \subseteq (B \otimes A) \wp (C \otimes A)$;

(i)* $(A \wp B) \wp C = A \wp (B \wp C)$;

(ii)* $A \wp \perp = A = \perp \wp A$;

(iii)* $A \subseteq B$ and $C \subseteq D$ implies $A \wp C \subseteq B \wp D$;

(iv)* $(A \oplus B) \oplus C = A \oplus (B \oplus C)$;

(v)* $A \oplus 0 = A = 0 \oplus A$;

(vi)* $A \subseteq A \oplus B$ and $B \subseteq A \oplus B$;

(vii)* $A \subseteq C$ and $B \subseteq C$ implies $A \oplus B \subseteq C$;

(viii)* $A \subseteq B$ and $C \subseteq D$ implies $A \oplus C \subseteq B \oplus D$;

(ix)* $A \oplus B = B \oplus A$;

(x)* $A \wp T = T = T \wp A$;

(xi)* $A \wp (B \& C) = (A \wp B) \& (A \wp C)$;

(xii)* $(B \& C) \wp A = (B \wp A) \& (C \wp A)$;

(xiii)* $(A \wp B) \otimes C \subseteq A \wp (B \otimes C)$;

(xiv)* $(A \wp B) \oplus (A \wp C) \subseteq A \wp (B \oplus C)$;

(xv)* $(B \wp A) \oplus (C \wp A) \subseteq (B \oplus C) \wp A$.

Proof: exercise.

1.4. PHASE SPACES WITH PREORDERS

Definition 1.4.1. A *preordered monoid* is a structure $(M, \bullet, u, \leq)$ where:

(i) $(M, \bullet, u)$ is a monoid;

(ii) $(M, \leq)$ is a preordered set (that is, $\leq$ is reflexive and transitive) ;

(iii) $\forall x, y, x', y' \in M$, if $x \leq y$ and $x' \leq y'$ then $x \bullet x' \leq y \bullet y'$ ($\bullet$ is monotone).

Examples 1.4.2. A partially *ordered group* is a structure $(G, \bullet, (-)^{-1}, u, \leq)$ where $(G, \bullet, (-)^{-1}, u)$ is a group, $(G, \leq)$ is a partially ordered set and the operation $\bullet$ is monotone. If G is commutative, then the ordered group is called *abelian*; if $(G, \leq)$ is a (conditionally complete) lattice, then the partially ordered group is called (complete) *l-group*. Any ordered group is an example of preordered monoid.

Definition 1.4.3. A preordered phase space is a structure $(M, \sigma, \perp, \leq)$ where:

(i) $(M, \sigma, \perp)$ is a non commutative phase space;

(ii) $(M, \leq)$ is a preordered monoid;

(iii) $\forall x, y \in M$, if $x \leq y$ and $y \in \perp$ then $x \in \perp$ (M is downward closed);

(iv) $\forall x,y \in M$, $\sigma(x \bullet y) = \sigma(x) \bullet \sigma(y)$, $\sigma(u)=u$;

(v) $\forall x,y \in M$, $x \leq y$ iff $\sigma(x) \leq \sigma(y)$.

If $(M, \perp)$ is a cyclic phase space, then the preordered phase space $(M, \perp, \leq)$ is called *cyclic phase space*; a cyclic phase space is called *preordered symmetric phase space* if $\forall$ x, $y \in M$, $x \bullet y \leq y \bullet x$ (*e-property*); a preordered symmetric phase space $(M, \perp, \leq)$ is called *preordered affine phase space* if $\forall x \in M$, $x \leq u$ (*w-property*); a preordered affine phase space is called *preordered classical phase space* if $\forall x \in M$, $x \leq x \bullet x$ (*c-property*). Note that in preordered phase spaces, for every pair of facts A and B:

$$(\forall a \in A\ \exists\, b \in B \text{ s.t. } a \leq b) \rightarrow A \subseteq^{\perp}B^{\perp}.$$

Example 1.4.4. Given a totally ordered abelian group $(G, \leq)$ and an element $x_0 \in G$ we posit: $\perp = \{ x : x \leq x_0\}$. Then, $(G, \perp, \leq)$ is preordered cyclic and symmetric phase space.

1.5. CANONICAL BASIS AND EXPONENTIALS

Definition 1.5.1. Given a preordered phase space $(M, \sigma, \perp, \leq)$ we indicate with **K** the following subset of M :

$\mathbf{K} = \{ x : \forall y \in M\ \ x \bullet y \leq y \bullet x \text{ and } y \bullet x \leq x \bullet y,\ x \leq u,\ x \leq x \bullet x \}$ (*canonical basis of M*).

Proposition 1.5.2. Given a preordered phase space $(M, \sigma, \perp, \leq)$ one has :

(i) **K** is a submonoid of M;

(ii) $\forall A \subseteq M$: $(A^{\perp} \cap \mathbf{K})^{\perp}$

(ii) $\forall x \in M$: $x \in \mathbf{K}$ iff $\sigma(x) \in \mathbf{K}$ iff $\sigma^{-1}(x) \in \mathbf{K}$;

(iii) $\forall A \subseteq M$: ${}^{\perp}(A^{\perp} \cap \mathbf{K}) = ({}^{\perp}A \cap \mathbf{K})^{\perp}$.

Proof : left to the reader.

Definition 1.5.3 (exponentials). Given a preordered phase space $(M, \sigma, \perp, \leq)$, we define the exponentials operators in the following way :

(i) $!A = {}^{\perp}(A \cap \mathbf{K})^{\perp}$;

(ii) $?A = {}^{\perp}(A^{\perp} \cap \mathbf{K}) = ({}^{\perp}A \cap \mathbf{K})^{\perp}$.

Proposition 1.5.4. In every preordered phase space $(M, \sigma, \perp, \leq)$, for every fact A we have that:

(i) $(!A)^{\perp} = ?A^{\perp}$;

(ii) ${}^{\perp}(!A) = ?{}^{\perp}A$;

(iii) $(?A)^{\perp} = !A^{\perp}$;

(iv) ${}^{\perp}(?A) = !{}^{\perp}A$.

Proof: (i) $(!A)^{\perp} = {}^{\perp}(A\cap\mathbf{K})^{\perp\perp} = (A\cap\mathbf{K})^{\perp} = ({}^{\perp}(A^{\perp})\cap\mathbf{K})^{\perp} = ?A^{\perp}$; (ii), (iii) and (iv): analogously.

Proposition 1.5.5. In every preordered phase space $(M, \perp, \leq)$, for every pair of facts A and B we have:

(i)	$A\subseteq B \;\rightarrow\; !A\subseteq !B$;	(i)*	$A \subseteq B \;\rightarrow\; ?A\subseteq ?B$;
(ii)	$!A\subseteq A$;	(ii)*	$A \subseteq ?A$;
(iii)	$!A\subseteq !!A$;	(iii)*	$??A \subseteq ?A$;
(iv)	$!A\otimes B = B\otimes !A$;	(iv)*	$?A \wp B = B \wp ?A$;
(v)	$!A\otimes !B\subseteq !(A\&B)$; $!T\subseteq 1$;	(v)*	$?(A\oplus B) \subseteq ?A \wp ?B$; $\perp\subseteq ?0$;
(vi)	$!(A\&B) \subseteq !A\otimes !B$; $1\subseteq !T$;	(vi)*	$?A \wp ?B\subseteq ?(A\oplus B)$; $?0\subseteq\perp$;
(vii)	$!A\subseteq 1$;	(vii)*	$\perp\subseteq ?A$;
(viii)	$!A\subseteq !A\otimes !A$;	(viii)*	$?A \wp ?A\subseteq ?A$.

Proof : exercise.

§2. ALGEBRAIC STRUCTURE OF THE SEQUENT CALCULUS FOR **NLL**.

In this section, the approach is close to the spirit of [2], [3], [13], [15] where syntactical objects and manipulations in relation to sequent calculus are handled in purely algebraic terms. The syntax of **NLL** is presented in the appendix. For a matter of convenience, structural rules are expressed in terms of sequence of formulas rather than, as usual, in terms of occurrences of formulas.

Proposition 2.1. The structure $(L^*, \bullet, e)$ where: L^* is the set of the sequences of formulas of L(**NLL**), $\bullet$ is the concatenation operation, and e is the empty sequence, is a monoid, called *syntactic monoid* of **NLL**.
Proof : Immediate.

Note also that the function: $(\text{-})^{2\perp} : L^* \longrightarrow L^*$ ($\mathbf{C} \longmapsto \mathbf{C}^{2\perp}$) is a bijection. Moreover, the structure $(L^*, (\text{-})^{2\perp}, \perp)$ is a non commutative phase space.

Definition 2.2. We posit $\perp = \{\mathbf{C}: \vdash\; \Rightarrow \mathbf{C^{op}}\}$, where $\mathbf{C^{op}} \equiv A_n, A_{n-1},\ldots, A_2, A_1$, if $\mathbf{C}$ is the sequence $A_1, A_2,\ldots, A_{n-1}, A_n$. $\perp$ is called *provability* of **NLL**.

Proposition 2.3. Let us consider the relation $\leq$ on $(L^*, \bullet, e)$ and posit $\mathbf{C} \leq \mathbf{D}$ when there exists in **NLL** a derivation λ from $\Rightarrow \mathbf{D}$ to $\Rightarrow \mathbf{C}$ in which only the rules ($?E^+$), ($?E^-$), (?W) and (?C) are applied. Then, $\leq$ is a preorder.

Proof : trivial.

Proposition 2.4. The structure $(L^*, (\_)^{2\perp}, \perp, \leq)$ is a preordered phase space and its canonical basis $\mathbf{K}$ is given by $\mathbf{K}=\{?\mathbf{D} : \mathbf{D} \in L^*\}$

Proof : Clearly, if $\mathbf{C} \in \mathbf{K}$, then $\mathbf{C} \leq e$. Then, $\mathbf{C}$ is obtained from the empty sequence by the application of the rules ($?E^+$), ($?E^-$), ($?W$) and ($?C$). It follows that $\mathbf{C} \equiv ?\mathbf{D}$, for some sequence $\mathbf{D}$. *Vice versa*, if $\mathbf{C} \equiv ?\mathbf{D}$, by means of the rules ($?E^+$), ($?E^-$), ($?W$) and ($?C$), one has that $\forall \mathbf{F} \in L^*$, $\mathbf{F}, \mathbf{C} \leq \mathbf{C}, \mathbf{F}$ and $\mathbf{C}, \mathbf{F} \leq \mathbf{F}, \mathbf{C}$; $\mathbf{C} \leq e$ and $\mathbf{C}, \mathbf{C} \leq \mathbf{C}$.

Definition 2.5. For every formula A, we posit : $PR(A) = \{\mathbf{C}: \vdash \Rightarrow \mathbf{C}^{op}, A\}$. PR(A) is called the *set of provability* of A.

Note that for every formula A, PR(A) is a fact. It is easy to show that:

(i) $PR(A \otimes B) = PR(A) \otimes PR(B)$;
(ii) $PR(A \& B) = PR(A) \& PR(B)$;
(iii) $PR(!A) = !PR(A)$;
(iv) $PR(1) = 1$;
(v) $PR(T) = T$;
(i)* $PR(A \wp B) = PR(A) \wp PR(B)$;
(ii)* $PR(A \oplus B) = PR(A) \oplus PR(B)$;
(iii)* $PR(?A) = ?PR(A)$.
(iv)* $PR(\perp) = \perp$;
(v)* $PR(0) = 0$.

Definition 2.6. A **NLL**-*Interpretation* is an ordered pair (S, α) where

(i) S is a preordered phase space;
(ii) $\alpha : VP_0 \longrightarrow Facts$

where VP_0 is the set of elementary propositional variables of L(**NLL**). One then defines by induction the function: $S^\alpha : L \longrightarrow Facts$

- $S^\alpha(p) = \alpha(p)$;
- $S^\alpha(p^{n+1\perp}) = (\alpha(p^{n\perp}))^\perp$;
- $S^\alpha(1) = 1$;
- $S^\alpha(T) = T$;
- $S^\alpha(B \otimes C) = S^\alpha(B) \otimes S^\alpha(C)$;
- $S^\alpha(B \& C) = S^\alpha(B) \& S^\alpha(C)$;
- $S^\alpha(!B) = !S^\alpha(B)$;
- $S^\alpha(p^{-(n+1)\perp}) = {}^\perp(\alpha(p^{-n\perp}))$;
- $S^\alpha(\perp) = \perp$;
- $S^\alpha(0) = 0$;
- $S^\alpha(B \wp C) = S^\alpha(B) \wp S^\alpha(C)$;
- $S^\alpha(B \oplus C) = S^\alpha(B) \oplus S^\alpha(C)$;
- $S^\alpha(?B) = ?S^\alpha(B)$.

A sequent $\Rightarrow \mathbf{C}$ is *valid* in (S, α) if:

i) $1 \subseteq S^\alpha(\wp \mathbf{C})$, if $\mathbf{C} \neq e$;
ii) $1 \subseteq \perp$, if $\mathbf{C} = e$.

A sequent $\Rightarrow \mathbf{C}$ is *valid* if it is valid in every **NLL**-interpretation. In this case, we write $\models \Rightarrow \mathbf{C}$.

Proposition 2.7. Let consider the **NLL**-*interpretation* (S, α) where:

(i) S is the preordered syntactic phase space of **AL**;
(ii) $\alpha : VP_0 \longrightarrow Facts$

Then, for every formula A, $S^\alpha(A) = PR(A)$.

Proof : by induction on the complexity of A.

Now we are in position to prove the routine soundness/completeness theorem.

Soundness theorem. For every sequent **C** of **NLL**, if $\vdash \Rightarrow$ **C** then $\models \Rightarrow$ **C**.
Proof : by induction on the number of applied rules in the proof of $\Rightarrow$ **C** by using the properties of the facts in the preordered phase space and the operations defined on them.

Completeness theorem. For every formula A of **NLL**, if $\models \Rightarrow$ A then $\vdash \Rightarrow$ A.

Let the formula A be such that $\models \Rightarrow$ A. Then, in particular, A is valid in the **NLL**-interpretation of Prop. 2.7. It readily follows that $1 \subseteq S^{\alpha}(A)$ and $PR(1) \subseteq PR(A)$. Finally, since $e \in PR(1)$, one has that $\vdash \Rightarrow$ A.

§3. CONCLUSION

It is evident in which sense the semantics for **NLL** is meant to be uniform over different but related logics (classical, affine, linear), determinated by the set of structural rules. Indeed, if we are in cyclic linear logic, we can consider the preordered *cyclic* phase space and its canonical basis **K**= {?**C**: **C**$\in L^*$}, if we are in linear logic, we can consider the preordered *symmetric* phase space and its canonical basis, and finally if we are in classical logic we can consider the preordered *classical* phase space (cf. Definition 1.4.3).

Classical sequent calculus without contraction and related semantics have been investigated in [6, 7] (see also Ono [10] for a comprehensive account of algebraic semantics for various "substructural logics"). More recently, Lafont [9] introduces a phase semantics for affine logic and proves soundness/completeness theorem by the standard method. The semantics he gives does not require any preorder. Nevertheless – as stressed for the first time by Castellan – the constrain of preorder is more convenient in the non commutative case and it has the advantage to providing a general context in which to approach to different, but close in spirit, logics. This kind of approach may be called *modular* since we can shift from non commutative linear logic (absence of all structural rules) to classical logic (presence of all structural rules), moving ourselves in the same semantical landscape. This obeys the methodological principle that a completeness theorem for related logics must have the same form. In other words, *semantics non facit saltus*.

APPENDIX

Definition 1. The language *L*(**NLL**) of the one-sided sequent calculus for **NLL** is defined as follows.

(i) The alphabet consists of the following symbols: propositional variables: $\{p^{n\perp}: n \in Z\}$. We will indicate with VP_0 the propositional variables of type: $p^{0\perp}$ (*elementary propositional*

variables). Propositional constant: $\perp$,1, 0, T; the binary connectives $\oplus$, $\wp$, &, $\otimes$; the exponentials !, ?; the sequent arrow $\Rightarrow$; and the usual auxiliary symbols.
(ii) Formulas are defined inductively as follows: every propositional variable is a formula; every propositional constant is a formula; if A and B are formulas, then $A\otimes B$, $A\wp B$, A&B and $A\oplus B$ are formulas; if A is a formula, then !A and ?A are formulas; nothing else is a formula.
(iii) The sequents are defined as follows: $\Rightarrow$ **C**, where is a finite sequence of formulas of the language.

Definition 2. For every formula A and for any $n \in Z$ the formula $A^{n\perp}$ is defined by induction on the complexity of A. For any $m \in Z$: $(p^{m\perp})^{n\perp} \equiv p^{m+n\perp}$;

if n =2k:	if n =2k+1:
$(1)^{2k\perp} \equiv 1$;	$(1)^{2k+1\perp} \equiv \perp$;
$(\perp)^{2k\perp} \equiv \perp$;	$(\perp)^{2k+1\perp} \equiv 1$;
$(T)^{2k\perp} \equiv T$;	$(T)^{2k+1\perp} \equiv 0$;
$(0)^{2k\perp} \equiv 0$;	$(0)^{2k+1\perp} \equiv T$;
$(B\otimes C)^{2k\perp} \equiv B^{2k\perp}\otimes C^{2k\perp}$;	$(B\otimes C)^{2k+1\perp} \equiv C^{2k+1\perp}\wp B^{2k+1\perp}$
$(B\wp C)^{2k\perp} \equiv B^{2k\perp}\wp C^{2k\perp}$;	$(B\wp C)^{2k+1\perp} \equiv C^{2k+1\perp}\otimes B^{2k+1\perp}$
$(B\&C)^{2k\perp} \equiv B^{2k\perp}\&C^{2k\perp}$;	$(B\&C)^{2k+1\perp} \equiv B^{2k+1\perp}\oplus C^{2k+1\perp}$
$(B\oplus C)^{2k\perp} \equiv B^{2k\perp}\oplus C^{2k\perp}$;	$(B\oplus C)^{2k+1\perp} \equiv B^{2k+1\perp}\&C^{2k+1\perp}$
$(!A)^{2k\perp} \equiv !A^{2k\perp}$;	$(!A)^{2k+1\perp} \equiv ?A^{2k+1\perp}$;
$(?A)^{2k\perp} \equiv ?A^{2k\perp}$;	$(?A)^{2k+1\perp} \equiv !A^{2k+1\perp}$.

We shall indicate the formulas $A^{1\perp}$ and $A^{-1\perp}$ with $A^{\perp}$ and $A^{-\perp}$ respectively. We observe that for every formula A: $A^{0\perp} = A$; and for every $m, n \in Z$: $(A^{m\perp})^{n\perp} \equiv A^{m+n\perp}$.

Definition 3. The one-sided sequent calculus for **NLL** is given by the following rules concerning the sequents of *L*(**NLL**).

Identity and cut rules:

$$\frac{}{\Rightarrow A^{\perp},\ A}\ (\text{ax}) \qquad \frac{\Rightarrow \mathbf{C},\ A \qquad \Rightarrow \mathbf{D},\ A^{\perp}}{\Rightarrow \mathbf{C}, \mathbf{D}}\ (\text{cut})$$

Cyclic rules:

$$\frac{\Rightarrow \mathbf{C}, \mathbf{D}}{\Rightarrow \mathbf{D}^{2\perp}, \mathbf{C}}\ ((\text{-})^{2\perp}) \qquad \frac{\Rightarrow \mathbf{C}, \mathbf{D}}{\Rightarrow \mathbf{D}, \mathbf{C}^{-2\perp}}\ ((\text{-})^{-2\perp})$$

Rules and axioms for the constants:

$$\frac{}{\Rightarrow 1}\ (1) \qquad \frac{\Rightarrow \mathbf{C}}{\Rightarrow \mathbf{C}, \bot}\ (\bot)$$

$$\frac{}{\Rightarrow \mathbf{C}, \mathrm{T}}\ (\mathrm{T})$$

Multiplicative logical rules:

$$\frac{\Rightarrow \mathbf{C}, A \qquad \Rightarrow \mathbf{D}, B}{\Rightarrow \mathbf{D}, \mathbf{C}, A \otimes B}\ (\otimes) \qquad \frac{\Rightarrow \mathbf{C}, A, B}{\Rightarrow \mathbf{C}, A \wp B}\ (\wp)$$

Additive logical rules:

$$\frac{\Rightarrow \mathbf{C}, A \qquad \Rightarrow \mathbf{C}, B}{\Rightarrow \mathbf{C}, A \,\&\, B}\ (\&) \qquad \frac{\Rightarrow \mathbf{C}, A}{\Rightarrow \mathbf{C}, A \oplus B}\ (\oplus 1) \qquad \frac{\Rightarrow \mathbf{C}, B}{\Rightarrow \mathbf{C}, A \oplus B}\ (\oplus 2)$$

Exponential structural rules:

$$\frac{\Rightarrow \mathbf{C}, \mathbf{E}}{\Rightarrow \mathbf{C}, ?\mathbf{D}, \mathbf{E}}\ (?W) \qquad \frac{\Rightarrow \mathbf{C}, ?\mathbf{D}, ?\mathbf{D}, \mathbf{E}}{\Rightarrow \mathbf{C}, ?\mathbf{D}, \mathbf{E}}\ (?C)$$

$$\frac{\Rightarrow \mathbf{C}, ?\mathbf{D}, \mathbf{E}, \mathbf{F}}{\Rightarrow \mathbf{C}, \mathbf{E}, ?\mathbf{D}, \mathbf{F}}\ (?E^{+}) \qquad \frac{\Rightarrow \mathbf{C}, \mathbf{D}, ?\mathbf{E}, \mathbf{F}}{\Rightarrow \mathbf{C}, ?\mathbf{E}, \mathbf{D}, \mathbf{F}}\ (?E^{-})$$

Exponential contextual and dereliction rules:

$$\frac{\Rightarrow ?\mathbf{C}, A}{\Rightarrow ?\mathbf{C}, !A}\ (!) \qquad \frac{\Rightarrow \mathbf{C}, A}{\Rightarrow \mathbf{C}, ?A}\ (?)$$

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