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Revisiting and Type-Freeing Church's Approach to Semantic Paradoxes

Pierdaniele Giaretta
Department of Philosophy, Sociology, Pedagogy and Applied Psychology
University of Padua
e-mail: pierdaniele.giaretta@unipd.it

ABSTRACT. We simplify and slightly modify the theory of types that Church provided with semantic primitive predicates. Two goals are pursued. The first goal is to present a simple application of Church's approach to paradoxes and to point out some aspects of this approach. The second, perhaps more interesting, goal is to show that when type distinctions are removed some basic Churchian principles need to be restricted and different restrictions correspond to Tarski's and Kripke's different approaches to truth. Finally, we briefly hint at how to move in the direction of Field's recent approach to truth by giving up some specific essential points of the Churchian framework.

KEYWORDS: Church, Paradox, Type, Semantic Value, Truth.

1. Introduction

From Bertrand Russell's perspective, paradoxes depend on a sort of linguistic inadequacy, which essentially consists in the failure to recognize some funda-

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mental ontological distinctions. If the relevant ontological distinctions are properly established and respected by the language, paradoxes cannot arise any more. Alonzo Church showed how the Russellian fundamental ontological distinctions should be applied to the relations of meaning between the linguistic expressions and the meant entities. Church's integration does not add anything basically new to Russell's classic approach. However, his integration is provided in a very clean and elegant formulation of Russell's type theory. Here we present a revision of Church's formulation and then free it from the Russellian ontological distinctions. Once we have omitted the type distinctions, some basic principles concerning the semantic relations are to be restricted, and different ways of conceiving and restricting them correspond to the different approaches to truth by Alfred Tarski and Saul Kripke.

Our starting point is Church's article "Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski", published on *The Journal of Symbolic Logic* in 1976. It was qualified a magisterial work, but because it falls outside of the main lines of research on truth and paradoxes it appears to be now considered just a beautiful piece of antique. Since we are convinced of the relevance of Church's approach beyond the Russellian framework which it belongs to, we'll change it partially so that it can more easily understood and connected with other major approaches to the theme of truth and paradoxes, mainly those of Tarski and Kripke.

We will introduce the following simplifying modifications. First, as concerns the language of Church's theory first expounded in Church (1976) and corrected in Church (1984), it is assumed that some individual constants are available which are to be taken as names for formulas. Second, a ternary val semantic predicate is replaced by a binary val semantic predicate. This modification is motivated by a correction introduced by Church (1984), but the binary val is not meant to serve all the purposes of the ternary val. In what may amount to a Russellian perspective (though not Russell's original perspective), this apparently strange feature can be used to express truth for sentences without resorting to propositions.

Third, a postulate-schema of Church (1976, 1984) is replaced with two different postulate schemata. The final aim is to separate two roles pertaining to the single postulate schema of Church (1976, 1984). If only simple type distinctions are adopted, a paradox only superficially similar to Russell's paradox is derivable. The derivation is simple since it only involves the quantification of properties and, in particular, it does not involve diagonalization. Even if it does not prove anything more than the Grelling paradox, it can demonstrate in a more direct way that the intuitive relation of being-true-of is paradoxical.

As is well known, the introduction of type ramification provides a special way of solving semantic paradoxes, and reducibility does not reintroduce them if distinctions of languages are not ignored. This is mentioned only briefly because both of these facts are based on results already well known from the literature.

Instead, we emphasize the modified Churchian framework independently of type distinctions. Our aim is to show that it is useful to analyze the source of the truth paradoxes. In particular, the different roles of the two kinds of Churchian postulates for the derivation of the Tarskian conditionals are pointed out. With reference to these two postulates, the Tarskian and the Kripkean perspectives can be accounted for as the consequences of two different theoretical choices that correspond to two general ways of conceiving the semantic values.

Existence and role of semantic values interact with logic. A specific interaction can be focused on with reference to our Churchian framework and provide a possible way to think of Field's radically different perspective.

2. From Church (1976) to Church (1984) and beyond

The language of Church (1976) and Church (1984) is a language built from typed variables and primitive constants. Church presents his language *L* with 'an unspecified list of primitive constants, each of definite r-type'; furthermore, he adds in note 5 that, 'it is intended that additions to the list of primitive constants may be made from time to time, so that Russell's formalized language is an open language rather than a language of fixed vocabulary' (Church 1976, p. 749). Such an openness of the language does appear to make the expressibility of its syntax relative to the various stages of the process of enlarging the list of the primitive constants. Indeed, Church does not care that his language expresses its own syntax: he only takes variables and formulas as values of variables for individuals and, since higher order quantification is available, that is enough to express a kind of self-reference that generates paradoxes. However, if 'additions to the list of primitive constants may be made from time to time' and formulas are values of variables for individuals, also being a value of a variable for individuals appear to be relative to a time. This feature is perhaps not so relevant until reducibility is taken into account.[1] It surely makes additional difficulties, if we as-

[1] See below. Some relativization would be needed in order to account for the denial of the expressibility of some concepts, which Church clearly admits.

sume the language is provided with names, precisely individual constants, for the expressions of the language, as we will do. Without denying that such difficulties can be successfully dealt with, let us give up the openness of the language for the sake of simplicity.

Introducing individual constants for the formulas of the language itself, which will allow us to state separately and easily the questions of the existence of a semantic value and of its constraints, is, however, a delicate matter. Some countable individual constants are bound to be interpreted as names of formulas by means of a suitable map such that every formula has a name and only one name. It follows that for every interpretation the domain of individuals is to be infinite. This limitation, which is already a consequence of Church's assumption that variables and formulas are values of variables for individuals, might be granted. However, how to fix a 1-1 function from some individual constants onto formulas? It would be easy, and dispensable, if the setting were so developed to include arithmetic. Without arithmetization available, we can presuppose some sort of systematic 1-1 coupling based on a list of individual constants and an enumeration of the formulas of the language. Identifying such a coupling with naming could be put into question, but we will not discuss the philosophical objections which might be raised when thinking that proper naming has to satisfy some special constraints.

The only constants introduced by Church are ternary val's. A val predicate applies to a variable, to a formula and to a property. Let us omit types and say that 'val(a, v, F)' means that a is a variable and v is a formula having no free variable other than a, and for every value x of the variable a the value of v is F(x). For simplicity, Church considers only the case that the variable a is for individuals and the variable F is for properties of individuals.

We will motivate the adoption of binary val's and we will remark that it suffices to define a notion of truth applying also to sentences in a way that is still Russellian. This modification can be presented and motivated independently of types. Even if we keep Church's notation referring to types in this section, the reader does not need to concern herself with them in this introductory section.

In Church's (1976) analysis of the Grelling paradox, the following definition of 'heterological' is provided:

$$\mathrm{het}^{n+1}(v) =_{df} \exists a \exists F^{1/n} (\mathrm{val}^{n+1}(a, v, F) \wedge \sim F(v))$$

and it is proved that

(6) $\forall x\, ((\mathrm{val}^{m+2}(a, v, G^{1/m+1}) \wedge G(x)) \leftrightarrow \mathrm{het}^{m+1}(x)) \rightarrow \sim \mathrm{het}^{n+1}(v)$, if $m \geq n$.

In the proof, there is the following passage (the second 'hence') that is not correct:

> Hence, by univ. inst. and P [propositional logic], $val^{m+2}(a, v, G)$, $val^{n+1}(a, v, F), \sim F(v) \vdash \sim G(v)$
> Hence, by ex. inst., $val^{m+2}(a, v, G), het^{n+1}(v) \vdash \sim G(v)$

The variable a is bounded by an existential quantifier in $het^{n+1}(v)$, on the left side of $\vdash$, but is free in the hypothesis $val^{m+2}(a, v, G)$. Given the extensional equivalence of F and G, $val^{2}(a, v, G)$ and $val^{1}(a, v, F)$, where m = 0, the flaw in the passage is shown by an interpretation of val (val^1 or val^2) such that for a true closed formula v and a specific individual variable x

$$val(a, v, F) \text{ is true if } \begin{cases} a = x \text{ and F is always true} \\ a \neq x \text{ and F is always false.} \end{cases}$$

An interpretation of this kind is no longer available after Church's (1984) correction of the proof of (6). Indeed, the correction amounts to the replacement of the postulate

(1) $(val^{m+1}(a, v, F^{1/m}) \wedge val^{n+1}(a, v, G^{1/n})) \rightarrow F = G$

from which it follows that

(2) $(val^{m+1}(a, v, F^{1/m}) \wedge val^{n+1}(a, v, G^{1/n})) \rightarrow \forall x\, (F(x) \leftrightarrow G(x))$

with the stronger postulate

$$(val^{m+1}(a, v, F^{1/m}) \wedge val^{n+1}(b, v, G^{1/n})) \rightarrow F = G$$

from which it follows that

$$(val^{m+1}(a, v, F^{1/m}) \wedge val^{n+1}(b, v, G^{1/n})) \rightarrow \forall x\, (F(x) \leftrightarrow G(x))$$

By adopting this postulate, it seems that the role of the first argument of val, which is meant to be an individual variable, becomes vacuous so that, instead of Church's ternary val, a binary val can be used, which applies to a formula v with at most one free variable and a property F: val(v, F).

It is worth remarking that, as happens in the case of Church's ternary val, nothing prevents val(v, F) from being true when v is a closed formula. By virtue of Church's postulate schema (3), adapted to the binary val,

(3) $\exists v \exists F\ (\mathrm{val}(v, F) \wedge \forall x\ (F(x) \leftrightarrow A))$

where A has at most the x variable as free variable and the predicate variable F is unary, we have that a sentence v expresses, relative to all individuals, a universal property if it is true and an empty property if it is false. A similar schema for a predicate val with a variable for binary relations as the second argument allows us to state that a true sentence expresses a universal binary relation and a false one an empty binary relation. Generalizing, by means of the appropriate val's and the corresponding schemata, one gets that a sentence expresses a property, a binary relation, a ternary relation, and so on, even if it is more natural to think that a sentence expresses a proposition—what can be said by a suitable postulate schema (3) for a predicate val that has a propositional variable as the second argument. No ambiguity ensues for the meaning of a sentence, since all these different semantic values are established by different meaning relations, expressed by different val predicates.

Of course, binary val's are not adequate to all purposes Church had in mind. We might need to refer to a the specific value of a specific variable, as, for example, to define Tarskian satisfaction relations. Informally, and omitting type distinctions, to define that formula v is satisfied by the values $x_1, x_2, ..., x_m$ of the variables $a_1, a_2, ..., a_m$, it should be said that there is a m-ary relation F such that $\mathrm{val}(a_1, a_2, ..., a_m, v, F)$ and $F(x_1, x_2, ..., x_m)$ (see Church 1984, p. 301). Even for a broader characterization of the semantic values of formulas, independently of the comparison with other approaches to semantics, reference to variables is surely relevant, as it appears from hints given by Church in fn 20 (see Church 1984, p. 299).

In the next two sections we are going to introduce a very simple paradox, which only superficially resembles Russell's paradox (therefore called the pseudo-Russell) to show how, analogously to the case of the Grelling paradox, a solution can be provided by introducing ramified types. Our aim it to point out some aspects Church's approach to paradoxes, particularly the need to keep separate type distinctions from expressibility matters. A theory of types where the type of propositions is omitted and only binary val predicates for properties are introduced is sufficient for our analysis.

3. Formal Statement and Derivation of the Pseudo-Russell in a Simple Type Theory

In the work of Church (1976) and (1984), ramified types are assigned to entities and to corresponding variables and constants. In order to emphasize the role of ramification as a way of solving antinomies, let us begin by neglecting it and keeping only the simple type distinctions. i is the type of individuals; $(\beta_1, ..., \beta_n)$, where $n \geq 1$ is the type of n-ary relations having, as arguments, entities of types $\beta_1, ..., \beta_n$ respectively. As already mentioned, and motivated as unnecessary to express a notion of truth for sentences, we omit the type of propositions (and, by extension, the variables and constants for propositions in the language).[2]

A corresponding language should be such that every variable and constant is assigned a simple type. Let us take into account a language *L* that, besides satisfying this minimal requirement, is provided with an infinite choice of variables for every type. An infinite amount of individual constants of type i is available. As anticipated, we suppose that they are such that it is possible to systematically assign some countable of them the role of names of formulas of *L*. Such an assignment is supposed to be known and will be used in the statement of the axioms and postulates.

Terms are variable or constants. An atomic formula is a sequence of symbols $F(t_1, ..., t_n)$, where $n \geq 1$, F is a variable or a primitive constant of a type $(\beta_1, ..., \beta_n)$ and $t_1, ..., t_n$ are variables or constants of the types $\beta_1, ..., \beta_n$ respectively.

Non-atomic formulas are built in the standard way, by means of all the usual connectives and quantifiers, i.e., without restricting to negation, disjunction and universal quantifier as in the work of Church and Russell.

Symbols, terms and formulas of *L* are taken to have the type i of individuals, so they can be values of (individual) variables and referred to by (individual) constants.

Suitable binary predicate constants val, here called val predicates, of various types are included in *L* in order to make it possible to 'speak' of the semantic values of formulas with at most one free variable.

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[2] As opposed to his earlier assumptions in Russell (1908), Russell (1910) took propositions as non-entities. Church was aware of this and defended the legitimacy, the coherence and, in a way, the need of propositions when propositional functions are endorsed. With reference to Church's considerations, Cocchiarella (1980) disagreed and remarked that it is technically possible to introduce the type hierarchy without the type of propositions.

In addition to strictly logical axioms and rules of inference appropriate for a typed language, the theory is provided with two kinds of axioms.

Comprehension axioms

1.1 $\exists F\, \forall x_1, ..., \forall x_n\, (F(x_1, ..., x_n) \leftrightarrow A)$

where $n \geq 1$, F is a relational variable of type $(\beta_1, ..., \beta_n)$, so that $x_1, ..., x_n$ are distinct variables of types $\beta_1, ..., \beta_n$ respectively, and F does not occur free in A.

Postulates for specific val *predicates*

val predicates are meant to express relations between a formula and a property.[3] Similar to the ternary val of Church (1984, p. 295), and omitting for simplicity type indications, a formula 'val(v, F)' is intended to mean that v is a formula having at most a free variable, and for every value x of the variable the value of v is F(x).

The postulates for the val predicates are, first of all, those expressing extensional univocacy (so-called by Church). Omitting for simplicity type indications, they have the following form:

2.1 $\forall v \forall F \forall G\, ((\mathrm{val}(v, F) \wedge \mathrm{val}(v, G)) \rightarrow \forall x\, (F(x) \leftrightarrow G(x)))$ *Univocacy*

Church's postulate schema (3) is replaced by two postulate schemata. While still omitting type indications, the instances of one postulate schema are as follows:

2.2 $\exists F\, \mathrm{val}([A], F)$ *Existence of semantic value*

where A is a formula in which at most one free variable of some type β occurs, [A] is the individual constant assigned to A and F is a 1-ary relational variable, not occurring free in A, of the type (β). The instances of the other postulate schema are as follows:

2.3 $\forall F\, (\mathrm{val}([A], F) \rightarrow \forall x\, (A \leftrightarrow F(x))$ *Semantic adequacy*

[3] F is taken by Church to be a propositional function, but we may take it as a property, leaving open the precise way in which it should be conceived.

where A is a formula in which at most one free variable of some type β occurs, [A] is the constant assigned to A, F is a 1-ary relational variable of type (β) not occurring free in A and 'x' is the variable occurring free in A, if any.

Then a relation for formulas with at most one free variable can be defined in order to represent the intuitive relation being-true-of of type (i, i):

$$T(v, x) =_{df} \exists P\, (val(v, P) \wedge P(x)).$$

T(v, x) is intended to be true when the formula v has a value—referred to by means of the 1-ary relational variable P—that is true of x.

By virtue of comprehension, T can be taken as a relation and not just as a tool for abbreviating the formula $\exists P\, (val(v, P) \wedge P(x))$. Then there is also a property R expressed by the formula $\sim T(x, x)$.[4] It is intuitively clear that R is paradoxical. Semi-formally, a proof can be given as follows:

Assume $\sim\exists P\, (val(r, P) \wedge P(r))$, where 'r' stands for $[\sim\exists P\, (val(x, P) \wedge P(x)]$. Then $\forall P\, (val(r, P) \rightarrow \sim P(r))$. By 2.2 there is a R such that val(r, R). $val(r, R) \rightarrow \sim R(r)$ follows from $\forall P\, (val(r, P) \rightarrow \sim P(r))$ by universal instantiation. Hence, by modus ponens, $\sim R(r)$. Keeping in mind that r is $[\sim\exists P\, (val(x, P) \wedge P(x)]$, by 2.3 we get $val(r, R) \rightarrow \forall x\, (\sim\exists P\, (val(x, P) \wedge P(x)) \leftrightarrow R(x))$. Thus, from val(r, R), by modus ponens, $\forall x\, (\sim\exists P\, (val(x, P) \wedge P(x)) \leftrightarrow R(x))$. Hence $\sim\exists P\, (val(r, P) \wedge P(r)) \leftrightarrow R(r)$. Since $\sim R(r)$, $\exists P\, (val(r, P) \wedge P(r))$.

Assume $\exists P\, (val(r, P) \wedge P(r))$, where r stands for $[\sim\exists P\, (val(x, P) \wedge P(x)]$. Then there is a P' such that $val(r, P')$ and $P'(r)$. By 2.3 $val(r, P') \rightarrow \forall x\, (\sim\exists P\, (val(x, P) \wedge P(x)) \leftrightarrow P'(x))$. Thus, by modus ponens, $\forall x\, (\sim\exists P\, (val(x, P) \wedge P(x)) \leftrightarrow P'(x))$. Hence $\sim\exists P\, (val(r, P) \wedge P(r)) \leftrightarrow P'(r)$. Since $P'(r)$, $\sim\exists P\, (val(r, P) \wedge P(r)$.

The paradox is apparently generated by the expression $\sim T(x, x)$, which looks like the formula that generates Russell's paradox. However, $\sim T(x, x)$ is a defined expression that stands for the formula $\sim\exists P\, (val(x, P) \wedge P(x))$, where differences of simple types are respected. Loosely speaking, the paradox shows the impossibility of taking val and $\sim\exists P\, (val(x, P) \wedge P(x)$ on a par with P. It

[4] A comparison with the notion of heterologicality might be useful. Heterological is defined by Church as $\exists P\, (val(x, P) \wedge \sim P(x))$ and turns out to be equivalent to $\sim T(x, x)$, i.e. $\sim\exists P\, (val(x, P) \wedge P(x))$, as a consequence of *Univocacy*, *Existence of semantic value*, and the intended meaning of val. However, $\sim T(x, x)$ looks (partially) analogous with Russell's paradox in a more direct way.

does not strictly depend on the interpretations of predicates and predicate variables as properties, even if the solution through ramification which we are going to consider introduces seemingly intensional distinctions.

Finally, it should be remarked that the derivation of $\sim\exists P\ (val(r, P) \wedge P(r))$ from $\exists P\ (val(r, P) \wedge P(r))$ does not contain any appeal to the *Existence of semantic value*. That might suggest that it is possible to avoid the paradox giving up only 2.2, without invalidating the second derivation and without diagnosing that the paradox depends on the simple type theory. However, ramification allows us to retain the *Existence of semantic value* (and *Semantic adequacy*) in forms such that the paradox is no longer derivable.

4. A solution through ramification

Once ramification is introduced, the above proof cannot be carried on any more. Let us define ramified types recursively, as Church does. From now on 'type' will stand for 'ramified type'.

i is a type and, if $\beta_1, ..., \beta_m$, where $m \geq 1$, are types, then $(\beta_1, ..., \beta_m)/n$, where $n \geq 1$, is a type. i is the type of individuals. $(\beta_1, ..., \beta_m)/n$ is the type of m-ary relations of level n having, as arguments, entities of types $\beta_1, ..., \beta_m$ respectively.

As with Church, let us define $(\alpha_1, ..., \alpha_m)/k < (\beta_1, ..., \beta_m)/n$ if $\alpha_1 = \beta_1, ..., \alpha_m = \beta_m$ and $k<n$. Entities of type α are intended to include entities of types $<\alpha$. Axioms should express cumulativity.

An order is assigned to every type in the following way: the order of the type i is 0, the order of a type $(\beta_1, ..., \beta_m)/n$ is N+n, where N is the greatest of the orders of the types $\beta_1, ..., \beta_m$.

Variables and constants of the language should be typed accordingly, and formulas should be constructed on the basis of the type distinctions. An atomic formula is a sequence of symbols $F(t_1, ..., t_m)$, where F is a variable or a primitive constant of a type $(\beta_1, ..., \beta_m)/n$, for some n and $m \geq 1$, and $t_1, ..., t_m$ are variables or constants of the types $\beta_1, ..., \beta_m$ respectively. Non-atomic formulas are built in the standard way. Let us keep the name '*L*' for the language so typed.

Let A be a well-formed formula. It is convenient to adopt the notion of the order of a variable or a constant identified, as usual, with the order of its type, and the notion of order of a formula: the order of a formula A, abbreviated by ord(A), is max(h, k+1), where h is the greatest order of free variables and constants occurring in A, and k is the greatest order of bound variables occurring in A.

Then the schema for comprehension axioms, here numbered as above, is modified in the following way:

1.1 $\exists F\ \forall x_1, ..., \forall x_m\ (F(x_1, ..., x_m) \leftrightarrow A)$

where F is a variable of type $(\beta_1, ..., \beta_m)/n$, $x_1, ..., x_m$ are distinct variables of types $\beta_1, ..., \beta_m$ respectively, $\mathrm{ord}(A) \leq \mathrm{ord}(F)$, and F does not occur free in A.

The postulates for val are to be suitably modified. Let us focus only on the second postulate schema for val.

The modification of 2.2, for formulas intuitively expressing properties of individuals, is as follows:

2.2 $\exists F\ \mathrm{val}^{n+1}([A], F)$ *Existence of semantic value*

where val^{n+1} has type (i, (i)/n)/1, A is a formula with at most one free variable of the type i, [A] is the constant assigned to A, $\mathrm{ord}(A) \leq n$ and F is a 1-ary variable of type (i)/n not occurring free in A.

To see how the above derivation of a paradox cannot be carried on any more, let us assume that 'x' has type i and 'P' and 'P′' have type (i)/1, so order 1, and are written as P^1 and as P'^1. val^{1+1} has order 2 and val^{2+1} has order 3. T(v, x), i.e. $\exists P^1\ (\mathrm{val}^{1+1}(v, P^1) \wedge P^1(x))$, has order 2. T, as a property, has order 2, indeed ≥ 2, according to the ramified version of 2.2, and, taking it as having order 2, is written as T^2. Similarly R, which is determined by $\sim\exists P^1\ (\mathrm{val}^{1+1}(x, P^1) \wedge P^1(x))$. Let us write R^2 for it. The outcome of such a specification of orders in the above derivation of the paradox is as follows, where the first half is reproduced only up to a clearly invalid inferential step:

Assume $\sim\exists P^1\ (\mathrm{val}^{1+1}(r, P^1) \wedge P^1(r))$, where r stands for $[\sim\exists P^1\ (\mathrm{val}^{1+1}(x, P^1) \wedge P^1(x))]$. Then $\forall P^1\ (\mathrm{val}^{1+1}(r, P^1) \rightarrow \sim P^1(r))$. By 2.2 there is a R^2 such that $(\mathrm{val}^{2+1}(r, R^2)$. $\mathrm{val}^{1+1}(r, R^2) \rightarrow \sim R^2(r)$ follows from $\forall P^1\ (\mathrm{val}^{1+1}(r, P^1) \rightarrow \sim P^1)(r))$ by universal instantiation. Hence, by modus ponens, $\sim R^2(r)$.

Assume $\exists P^1\ (\mathrm{val}^{1+1}(r, P^1) \wedge P^1(r))$, where r stands for $[\sim\exists P^1\ (\mathrm{val}^{1+1}(x, P^1) \wedge P^1(x))]$. Then there is a P'^1 such that $\mathrm{val}^{1+1}(r, P^1)$ and $P'^1(r)$. By 2.3 $(\mathrm{val}^{1+1}(r, P'^1) \rightarrow \forall x\ (\sim\exists P^1\ (\mathrm{val}^{1+1}(x, P^1) \wedge P^1(x)) \leftrightarrow P'^1(x))$. Thus, by modus ponens, $\forall x\ (\sim\exists P^1\ (\mathrm{val}^{1+1}(x, P^1) \wedge P^1(x)) \leftrightarrow P'^1(x))$. Hence $\sim\exists P^1\ (\mathrm{val}^{1+1}(r, P^1) \wedge P^1(r)) \leftrightarrow P'^1(r)$. Since $P'^1(r)$, $\sim\exists P^1\ (\mathrm{val}^{1+1}(r, P^1) \wedge P^1(r))$.

While the first part of the derivation is invalid, nothing violates the ramification distinctions in the latter part of the derivation. This provides a proof for $\sim\exists P^1 (val^{1+1}(r, P^1) \wedge P^1(r))$, shortly $\sim T^2(r, r)$ or $R^2(r)$. In fact, no property of order 1 can be expressed by the formula $\sim\exists P^1 (val^{1+1}(x, P^1) \wedge P^1(x))$.

However, as is well known, reducibility axioms, whose merits or defects will not be discussed here, allow us to replace the commitment to high-level entities with a commitment to lower-level entities. In particular, according the appropriate reducibility axiom, there is a R^1 such that

$$\forall x(R^1(x) \leftrightarrow R^2(x)).$$

It follows that $\forall P^1 (val^{1+1}(r, P^1) \rightarrow \sim P^1(r))$ admits the following instance:

$$val^{1+1}(r, R^1) \rightarrow \sim R^1(r).$$

The antecedent $val^{1+1}(r, R^1)$ of this implication cannot be asserted or consistently assumed. If $val^{1+1}(r, R^1)$ is assumed, the paradox is derivable by replacing R^2 with R^1 and the derivation just reduces the assumption to absurdum. Thus $\sim val^{1+1}(r, R^1)$.

On the other hand, one might think that reducibility legitimizes the introduction of a constant TR^1 for a R^1 extensionally equivalent to R^2. Then the formula $TR^1(x)$ could be used to derive the paradox. However, it would be a natural reaction to conclude that such a constant for R^1 cannot belong to the language *L* and reducibility cannot legitimize its introduction in the language *L*.[5]

5. Removing Types and Connecting Principles for val and true-of

When types are removed, the postulates of the forms 2.1, 2.2 and 2.3 do not lose their interest. They can be appreciated because of the insights they provide into possible type-free treatments of semantic paradoxes, and more specifically because of their apparent relations with the Tarskian conditionals and the ways these conditionals might be restricted to avoid the paradoxes in a classical context.

[5] All this is more or less implicit in remarks first made by Church (1976) and Myhill (1979).

Let us state again 2.1, 2.2 and 2.3, but without any type distinctions except the distinction between individual and predicate variables:

2.1 $\forall v \forall F \forall G\ (\mathrm{val}(v, F) \wedge \mathrm{val}(v, G) \rightarrow \forall x\ (F(x) \leftrightarrow G(x))$ *Univocacy*
2.2 $\exists F\ \mathrm{val}([A], F)$ *Existence of semantic value*

where A is a formula with at most one free variable, [A] is the constant assigned to A and F is a 1-ary predicate variable not occurring free in A.

2.3 $\forall F\ (\mathrm{val}([A], F) \rightarrow \forall x\ (A \leftrightarrow F(x))$ *Semantic adequacy*

where A is a formula with at most one free variable, [A] is the constant assigned to A, F is a 1-ary predicate variable not occurring free in A and 'x' is the individual variable occurring free in A, if any.

With type distinctions omitted, the paradox here considered, like many other paradoxes of various kinds, is derivable and it is clear that semantic closeness, as expressed by 2.2 (*Existence of semantic value*) and 2.3 (*Semantic adequacy*) for the val predicates, is required to derive it.

However, 2.2 and 2.3 have very different roles. Giving up only 2.2 does not block the derivation of $\sim\exists P\ (\mathrm{val}(r, P) \wedge P(r))$ from $\exists P\ (\mathrm{val}(r, P) \wedge P(r))$, where r stands for $[\sim\exists P\ (\mathrm{val}(x, P) \wedge P(x))]$, i.e., the proof of

(*) $\sim\exists P\ (\mathrm{val}(r, P) \wedge P(r))$

or, equivalently,

$$\forall P\ (\mathrm{val}(r, P) \rightarrow \sim P(r)).$$

If the postulates 2.2 are given up, and it is assumed that there is a R such that

(abs) $\mathrm{val}(r, R) \wedge \forall x\ (R(x) \leftrightarrow \sim\exists P\ (\mathrm{val}(x, P) \wedge P(x)))$

we get from $\forall P\ (\mathrm{val}(r, P) \rightarrow \sim P(r))$

$$\mathrm{val}(r, R) \rightarrow \sim R(r)$$

hence, from (abs)

$$\sim R(r)$$

so

$$\exists P\ (\mathrm{val}(r, P) \wedge P(r))$$

against (*). Thus, by reduction

$$\sim\exists R\ (\mathrm{val}(r, R) \wedge \forall x\ (R(x) \leftrightarrow \sim\exists P\ (\mathrm{val}(x, P) \wedge P(x))).$$

All this can look very familiar because of the similarity with the Grelling paradox, which we will briefly take into account in the context of our reformulation of Church's theory.

As is well known, this paradox is obtained by considering the formula

$$\exists P\ (\mathrm{val}(v, P) \wedge \sim P(v)).$$

Let us take 'het' as the constant for $\exists P\ (\mathrm{val}(v, P) \wedge \sim P(v))$. It is easy to use 2.3 to derive $\sim\exists P\ (\mathrm{val}(het, P) \wedge \sim P(het))$ from $\exists P\ (\mathrm{val}(het, P) \wedge \sim P(het))$. On the other hand, suppose $\sim\exists P\ (\mathrm{val}(het, P) \wedge \sim P(het))$. Then, for every P, if val(het, P), then P(het). By 2.2, for some P′, val(het, P′). Hence P′(het). By 2.3, $\forall x\ (\exists P\ (\mathrm{val}(x, P) \wedge \sim P(x)) \leftrightarrow P'(x))$. Thus $\exists P\ (\mathrm{val}(het, P) \wedge \sim P(het))$ by instantiation and propositional logic. As is the case above, if 2.3. is accepted and 2. 2 is given up, and it is assumed that there is a H such that

$$\mathrm{val}(het, H) \wedge \forall x\ (H(x) \leftrightarrow \sim\exists P\ (\mathrm{val}(x, P) \wedge \sim P(x)))$$

the derivation of a contradiction proves

$$\sim\exists H\ (\mathrm{val}(het, H) \wedge \forall x\ (H(x) \leftrightarrow \sim\exists P\ (\mathrm{val}(x, P) \wedge P(x))).$$

2.3 (*Semantic adequacy*) is resorted to in the proofs of both directions of the pseudo-Russell and of the Grelling paradox. It has a major role in expressing the characteristic features of the semantic relation val, whereas 2.2 (*Existence of semantic value*) just states a sort of universal applicability of val, even to formulas containing val, and hence implies a sort of semantic closure of the language which val belongs to.

The different roles of *Existence of semantic value* and *Semantic adequacy* are highlighted when they are connected with the Tarskian conditionals for the notion of true of. Let us take this notion as represented by T(v, w), according to the above adopted and here repeated definition:

$$T(v, w) =_{df} \exists F\ (val(v, F) \wedge F(w))$$

where F is a 1-ary relational variable. It is quite natural to assume that T fulfils the instances of the following schemata:

R $\quad T([A], x) \rightarrow A$
C $\quad A \rightarrow T([A], x)$

where [A] is a formula with at most one free variable and x is such a variable, if any. Recently, the corresponding principles for the truth predicate have been labeled respectively 'Release' and 'Capture'.[6] Here the letters 'R' and 'C' can be used in a similar sense.

It is very easy to derive R and C from 2.2 and 2.3:

Proof of R
Let us assume $T([A], x)$, i.e., $\exists F\ (val([A], F) \wedge F(x))$. So, for some F', $val([A], F') \wedge F'(x)$ and, by 2.3, $val([A], F') \rightarrow \forall x\ (A \leftrightarrow F'(x))$. Thus $A \leftrightarrow F'(x)$ and, since $F'(x)$, then A.

Proof of C
Let us assume A. By 2.2, $\exists F\ val([A], F)$. So, for some F', $val([A], F')$. By 2.3, $val([A], F') \rightarrow \forall x\ (A \leftrightarrow F'(x))$. Thus $A \leftrightarrow F'(x)$ follows. Since A by assumption, $F'(x)$. Then $val([A], F') \wedge F'(x)$. So $\exists F\ (val([A], F) \wedge F(x))$, i.e., $T([A], x)$.

It should be noted that 2.2 (*Existence of semantic value*) is utilized only in the proof of C, whereas 2.3 (*Semantic adequacy*) has a role in both the proof of R and the proof of C.

R and C together provide the Tarskian biconditionals. We should note that, in addition to R, there is also a restricted version of the Tarskian biconditionals that does not depend on 2.2. One such version is stated by Kripke for the closed off interpretation of the truth predicate T at the minimal fixed point of his semantic hierarchy (Kripke, 1975) and is informally introduced by Parsons (1974). Kripke's restricted version of the Tarskian biconditionals is stated for

[6] See, as an example, Beall and Glanzberg (2011). Indeed, Beall and Glanzberg (2011) do not introduce R and C with '→', but with '|–' taken as a place-holder for—they say—"a range of different logical notions, each of which will provide some notion of valid inference in some logical theory". Thus our R and C, stated above as valid classic implications, are specific principles instantiating their schemata.

an *undefined* truth predicate of sentences of a first order interpreted language as follows:

K $(T([A]) \vee T([\sim A])) \rightarrow (A \leftrightarrow T([A]))$

val 2-nary predicates of type (i, n/0), where n/0 is a type of propositions, can be introduced in a Churchian language, so that T predicates, specific for sentences, can be defined in the following natural way:

$$T(v) =_{df} \exists p\ (val(v, p) \wedge p)$$

where v is an individual variable and p is a propositional variable of type n/0. If propositions are taken to be of the same type, only one T predicate specific for sentences is defined in the above way. However, we can dispense with propositions and propositional variables, and go on with our true-of predicate expressed by T(v, w) as previously defined and state K as:

K′ $(T([A], x) \vee T([\sim A], x) \rightarrow (A \leftrightarrow T([A], x))$

K′ is easily derivable from 2.3.

Proof

Let us assume $T([A], x) \vee T([\sim A], x)$, i.e., $\exists F\ (val([A], F) \wedge F(x)) \vee \exists F\ (val([\sim A], F) \wedge F(x))$. Let us prove that each disjunct entails $A(x) \leftrightarrow T([A], x)$, i.e., $A \leftrightarrow \exists F\ (val([A], F) \wedge F(x))$:

1. $\exists F\ (val([A], F) \wedge F(x))$ (hyp.)
   By propositional logic, $A \rightarrow \exists F\ (val([A], F) \wedge F(x))$. On the other hand, by hyp., for some F′, val([A], F′) and F′(x). By 2.3 $A \leftrightarrow F'(x)$. So A. Thus $\exists F\ (val([A], F) \wedge F(x)) \rightarrow A$.
2. $\exists F\ (val([\sim A], F) \wedge F(x))$ (hyp.)
   For some F′, val([~A], F′) and F′(x). By 2.3 $\sim A \leftrightarrow F'(x)$; thus ~A. By propositional logic, $A \rightarrow \exists F\ (val([A], F) \wedge F(x))$. Concerning the other direction, assume $\exists F\ (val([A], F) \wedge F(x))$. Then for some F′, val([A], F′) and F′(x). By 2.3 $A \leftrightarrow F'(x)$; thus A, against the previous derivation of ~A. Thus $\sim\exists F\ (val([A], F) \wedge F(x))$. By propositional logic, $\exists F\ (val([A], F) \wedge F(x)) \rightarrow A$.

Vice versa, is 2.3 (*Semantic adequacy*) derivable from K′? It is, if the following principle is adopted:

NEG $\quad val([A], F) \to \exists G\ (val([\sim A], G) \wedge \forall x\ (G(x) \leftrightarrow \sim F(x)))$.

Loosely speaking, NEG says that if A expresses F, then ~A expresses ~F. It allows the following derivation of 2.3 from K′.

Proof
Let us assume val([A], F*) (hyp.). It will be proved that $A \leftrightarrow F^*(x)$, where x is the only variable occurring in A, if any.

1. Suppose F*(x). Then, by hyp., $\exists F\ (val([A], F) \wedge F(x))$, i.e., T([A], x). By K′, A. Thus $F^*(x) \to A$.
2. Suppose A. Let us take into account the cases T([A], x), T([~A], x), $\sim T([A], x) \wedge \sim T([\sim A], x)$.

2a. T([A], x), i.e. $\exists F\ (val([A], F) \wedge F(x))$. So, for some F′, val([A], F′) and F′(x). By hyp. and 2.1, $F^*(x) \leftrightarrow F'(x)$, hence F*(x).
2b. T([~A], x). By K′ and supposition 2, T([A], x). So, for some F′, val([A], F′) and F′(x). By 2.1, $F^*(x) \leftrightarrow F'(x)$, hence F*(x).
2c. $\sim T([A], x) \wedge \sim T([\sim A], x)$. So (i) $\forall F\ (val([A], F) \to \sim F(x))$ and (ii) $\forall F\ (val([\sim A], F) \to \sim F(x))$. By hyp. and (i), ~F*(x). By hyp. and NEG, for some G, val([~A], G) and $G(x) \leftrightarrow \sim F^*(x)$. By (ii), ~G(x). Hence, F*(x). Thus, by reductio, hyp. is false.

It follows that

$$val([A], F) \to (F(x) \to A)$$
$$val([A], F) \to (A \to F(x))$$

whence 2.3.

Thus, under the quite natural assumption NEG, K′ is equivalent to *Semantic adequacy*. Since *Semantic adequacy* entails R, it follows that Kripke (1975) is committed to R.[7]

As well known, Kripke's attitude towards R and C is not Tarskian. The difference can be clearly accounted for within Church's framework.

Let us suppose that A has no free variable, i.e., A is a sentence and x is an individual variable. Then

R $\quad T([A], x) \to A$

says that if A expresses a property true of an individual, then A (or A is intuitively true), and

[7] The endorsement of R by Kripke (1975) was first, in a different way, shown by Feferman (1984, pp. 101-102).

C $\quad A \rightarrow T([A], x)$

says that if A (or A is intuitively true), A expresses a property true of an individual.

Surely this is not the meaning assigned to Tarskian conditionals either by Tarski or Kripke. However, R and C are together contradictory, like the intuitive conditionals taken into account by Tarski and Kripke. Their derivation from the more basic Churchian principles 2.2. and 2.3 suggest the following account of the different ways in which Tarski and Kripke pursue the goal of avoiding the semantic paradoxes.

Their different approaches may be traced to different imaginary choices concerning the validity of the val principles. Assuming that Kripke's final move is the closing off, our fiction is as follows. Both Tarski and Kripke endorse 2.1 (*Univocacy*) without any limitation. Both require 2.3 (*Semantic adequacy*) to hold. However, Tarski also requires that 2.2 (*Existence of semantic value*) be generally valid, in order for his biconditionals for truth to hold. So, to avoid the contradiction, he has to restrict the range of formulas upon which the val relation, and therefore truth, is defined. Kripke appears to give up the general validity of 2.2 (*Existence of semantic value*), while allowing—consistently with the closing off—that the val relation is defined on formulas having no semantic value. It follows that the antecedent of 2.3 (*Semantic adequacy*) is false for some formulas and not all Tarskian biconditionals can be asserted.

6. A Final Look: How to Move in the Direction of Field's Recent Approach

The predicate T of a Liar sentence is usually undefined. Kripke's Liar form is $\forall x\ (P(x) \rightarrow {\sim}T(x))$, where P(x) is a syntactic predicate uniquely satisfied by the code of $\forall x\ (P(x) \rightarrow {\sim}T(x))$. A shorter standard form is ${\sim}T(l)$, where l is a term whose value is the formula ${\sim}T(l)$. When T(x) is understood as saying that x expresses a property true of any individual, Kripke's semantic construction can be conceived as a way of systematically identifying the sentences expressing a universal property. However, the specific kind of values that may be assigned to sentences is not so relevant. The outcome matters: an interpretation of T is reached at the minimal fixed point such that some sentences do not come out as true and their negations either. A Liar sentence is such a sentence. By means of the closing off it can be acknowledged as intuitively true but is not true according to the fixed point interpretation of T. Thus, as emphasized by Field, both the Liar and the negation that the Liar is true should be asserted.

This awkward result essentially depends on the non-validity of bivalence —which, using the Churchian true-of predicate, is expressed by means of the schema $T([A], x) \vee T([\sim A], x)$—and on the final preservation of classical logic. In our Churchian framework bivalence follows from *Existence of semantic value*, *Semantic adequacy*, NEG and Excluded Middle. For, by *Existence of semantic value*, both $A \rightarrow \exists F\ val([A], F)$ and $\sim A \rightarrow \exists F\ val([\sim A], F)$. Then, given A, by *Semantic adequacy*, $\exists F\ (val([A], F) \wedge F(x))$, and, given ~A, by *Semantic adequacy* and NEG, $\exists F\ (val([\sim A], F) \wedge \sim F(x))$. By Excluded Middle, $\exists F\ (val([A], F) \wedge F(x)) \vee \exists F\ (val([\sim A], F) \wedge \sim F(x))$, i.e. $T([A], x) \vee T([\sim A], x)$. From the outlined Churchian perspective, Kripke's approach amounts to giving up *Existence of semantic value*, in agreement with his informal introductory remarks arguing for the compatibility of linguistic legitimacy and lack of semantic value.

However, another possible move is giving up Excluded Middle. Then acknowledging the lack of designated value is not a sufficient reason for asserting the negation and a correlate move is to introduce a conditional allowing the restatement of Tarskian biconditionals in such a way that they come out true. This is the perspective that Field develops in great detail. Of course, the abandonment of classical logic and—we should add—the rejection of ontological commitment to semantic values locate it completely outside a Churchian framework.[8]

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