Set Graphs. V. On representing graphs as membership digraphs

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Abstract

An undirected graph is commonly represented as a set of vertices and a set of doubletons of vertices; but one can also represent vertices by finite sets so as to ensure that membership mimics, over those sets, the edge relation of the graph. This alternative modeling, applied to connected claw-free graphs, recently gave crucial clues for obtaining simpler proofs of some of their properties (e.g., Hamiltonicity of the square of the graph).

This paper adds a computer-checked contribution. On the one hand we discuss our development, by means of the Ref verifier, of two theorems on representing graphs by families of finite sets: a weaker theorem pertains to general graphs, and a stronger one to connected claw-free graphs. Before proving those theorems, we must show that every graph admits an acyclic, weakly extensional orientation, which becomes fully extensional when connectivity and claw-freeness are met. This preliminary work enables injective decoration, *à la* Mostowski, of the vertices by the sought-for finite sets. By this new scenario, we complement our earlier formalization with Ref of two classical properties of connected claw-free graphs. On the other hand, our present work provides another example of the ease with which graph-theoretic results are proved with the Ref verifier. For example, we managed to define and exploit the notion of connected graph without resorting to the notion of path.

Key words: Theory-based automated reasoning; proof checking; Referee aka ÆtnaNova; graphs and digraphs; Mostowski's decoration.

1 Can graphs be represented as membership digraphs?

One usually views the edges of a graph[1] as vertex doubletons; but various ways of representing graphs can be devised (as quickly surveyed in [10, end of Sec. 2]). Thanks to a convenient choice on how to represent *connected claw-free graphs*,[2] Milanič and Tomescu [5] proved with relative ease two classical propositions on graphs of that kind, namely that any such graph owns a near-perfect matching and has a Hamiltonian cycle in its square; a proof of the somewhat deeper theorem [4] that all connected claw-free graphs have a *vertex-pancyclic* square was also attained cheaply through the same representation [13]. Specifically, the facilitation stems from transferring those results to the special class of the *membership digraphs*, whose set of vertices is a hereditarily finite set and whose arcs precisely reflect the membership relation between vertices. Under this change of perspective, a fully formal reconstruction of the first two results became affordable and, once carried out, was certified correct with the Ref proof-checker [8, 9, 10].

Can we, with equal ease, formalize in Ref the Milanič-Tomescu representation result *per se*? This paper provides a positive answer, thus achieving one of the continuations of [10] envisaged in [9, Sec. A.10].

We started with a wide-scope formalization task, by proving with Ref that a graph G whatsoever admits a set ν_G of finite sets and an injection f from the vertices of G onto ν_G such that $\{x, y\}$ is an edge of G if and only if either $fx \in fy$ or $fy \in fx$ holds. The proof articulates as follows:

[1] We call undirected graphs simply *graphs*, and directed graphs, *digraphs*.

[2] A graph is said to be *claw-free* if no induced subgraph of its is isomorphic to the graph $K_{1,3}$, called *the claw*, depicted in Fig. 14 of this paper.

1. For any $G = (V, E)$, there is a $D \subseteq V \times V$ s.t. $E = \big\{\{x,\ y\} : [x, y] \in D\big\}$ and (V, D) is an *acyclic* digraph which is *weakly extensional*: i.e., any two vertices that share the same out-neighbors have *no* out-neighbors.

2. We injectively *decorate* vertices by putting $f\,v = \{f\,w : [v, w] \in D\}$ for each $v \in V$ endowed with out-neighbors, $f\,z = \emptyset$ for one sink z, and by assigning suitable non-null values $f\,u$ to all sinks $u \neq z$ in order that weak extensionality ensure the injectivity of f. Note that acyclicity ensures that the recursive characterization of f makes sense.

The more specific Milanič–Tomescu representation theorem insists, for a connected claw-free graph G, on the condition $\bigcup \nu_G \subseteq \nu_G$, which is crucial in the exploitation of the theorem. The new condition means *transitivity*, i.e. that $x \in y \in \nu_G$ must imply $x \in \nu_G$; moreover, it implies that ν_G is *hereditarily* finite. The proof now articulates as follows:

1′. One shows that for any graph $G = (V, E)$ as said, there is a $D \subseteq V \times V$ such that $E = \big\{\{x,\ y\} : [x, y] \in D\big\}$ and (V, D) is an acyclic digraph which is *extensional*: i.e., no two vertices in V have the same out-neighbors.

2′. One decorates vertices by putting $f\,v = \{f\,w : [v, w] \in D\}$ *à la* Mostowski, for all $v \in V$. Extensionality ensures the injectivity of f.

(Notice that 2. subsumes 2′. altogether, because an extensional digraph has exactly one sink.)

It was proved in [5] that other classes of graphs admit such a representation by hereditarily finite sets, for example graphs with a Hamiltonian path. However, it is an NP-complete problem to decide in full generality whether a given graph G admits such a transitive set ν_G [6]. The inductive proof of 1′. that will be followed in our formalization task offered in this paper is actually a simplification of the original proof in [5], one that also leads to a linear-time algorithm for constructing the transitive set ν_G [7]. Moreover, the fact that the class of claw-free graphs is the largest class of graphs, closed under taking induced subgraphs, with the property that every connected member G of it admits such a transitive ν_G (since the claw does not admit one), makes this representation theorem rather worthy. Further evidence of the close kinship between connected claw-free graphs and membership digraphs comes from the observation that the transitivity property is actually crucial to obtain the two simple proofs presented in [8, 9, 10]. For example, since the removal of an $\in$-maximal element from a transitive set ν_G leads to another transitive set, this representation gives an immediate hook for inductive arguments (see the details in [10]).

The proof-checking experiment embodying 1. and 2. is discussed in Section 3, after a glimpse of the main features of the Ref system in Section 2. Then we move on to a discussion on the more engaging experiment embodying 1′. and 2′.—also carried out with Ref—in Section 4.

The experiments on which we will report are available at `http://www2.units.it/eomodeo/GraphsViaMembership.html`. They contain 30 definitions and 109 theorems, organized in 8 THEORYs. The overall number of proof lines is 1818, there are 5 proofs whose length exceeds 50 lines (the highest length being 73), and processing the entire scenario takes approximately 15 seconds.

2 Some ingredients of our Ref scenario

While referring the interested reader to [10, Sec. 3] for more detailed information of the Ref proof checker, here we briefly illustrate its formalism with examples taken from the experiment on which we are reporting.

What one submits to Ref, to have its correctness verified, is a SCENARIO: namely, a script file consisting of definitions and of theorems endowed with their proofs; a construct, named THEORY, enables one to package definitions and theorems into reusable proofware components. A variant of the Zermelo-Fraenkel set theory, postulating global choice, regularity, and infinity, underlies the logical armory of Ref: this is apparent from the fifteen or so inference rules available in the proof-specification language, of which only a few sprout directly from first-order predicate calculus, while most embody some form of set-theoretic reasoning. Multi-level syllogistic [3] acts as a ubiquitous inference mechanism, while THEORYs add a touch of second-order reasoning ability to Ref's overall machinery.

Our initial figures offer a glimpse of the Ref's language. Fig. 1 shows the definitions of graph-theoretic notions relevant to the proof-checking experiment on which we report, and introduces the notions of

DEF acyclic: [Acyclicity] $\mathsf{Acyclic}(\mathsf{V},\mathsf{D}) \leftrightarrow_{\mathrm{Def}}$

$$\left\langle \forall \mathsf{w} \subseteq \mathsf{V} \mid \mathsf{w} \neq \emptyset \rightarrow \left\langle \exists \mathsf{t} \in \mathsf{w} \mid \emptyset = \{\mathsf{y} \in \mathsf{w} \mid [\mathsf{t},\mathsf{y}] \in \mathsf{D}\} \right\rangle \right\rangle$$

DEF xtens$_0$: [Extensionality] $\mathsf{Extensional}(\mathsf{V},\mathsf{D}) \leftrightarrow_{\mathrm{Def}}$

$$\left\langle \forall \mathsf{x} \in \mathsf{V}, \mathsf{y} \in \mathsf{V}, \exists \mathsf{z} \mid ([\mathsf{x},\mathsf{z}] \in \mathsf{D} \leftrightarrow [\mathsf{y},\mathsf{z}] \in \mathsf{D}) \rightarrow \mathsf{x} = \mathsf{y} \right\rangle$$

DEF xtens$_1$: [Weak extensionality] $\mathsf{WExtensional}(\mathsf{V},\mathsf{D}) \leftrightarrow_{\mathrm{Def}}$

$$\mathsf{Extensional}\big(\mathsf{V} \cap \mathbf{dom}(\mathsf{D} \cap (\mathsf{V} \times \mathsf{V})), \mathsf{D} \cap (\mathsf{V} \times \mathsf{V})\big)$$

DEF orien : [Orientation of a graph] $\mathsf{Orientates}(\mathsf{D},\mathsf{V},\mathsf{E}) \leftrightarrow_{\mathrm{Def}}$

$$\mathsf{E} \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{V}, \mathsf{y} \in \mathsf{V} \setminus \{\mathsf{x}\}\} = \left\{\{\mathsf{p}^{[1]},\mathsf{p}^{[2]}\} : \mathsf{p} \in \mathsf{D} \mid \mathsf{p} = [\mathsf{p}^{[1]},\mathsf{p}^{[2]}]\right\}$$

DEF maps$_1$: [Map domain, i.e. first components of pairs in map] $\mathbf{dom}(\mathsf{F}) =_{\mathrm{Def}}$

$$\{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{F}\}$$

DEF maps$_2$: [Map restriction] $\mathsf{F}_{|\mathsf{A}} =_{\mathrm{Def}}$

$$\{\mathsf{p} \in \mathsf{F} \mid \mathsf{p}^{[1]} \in \mathsf{A}\}$$

DEF maps$_3$: [Value of single-valued function] $\mathsf{F}\upharpoonright\mathsf{X} =_{\mathrm{Def}}$

$$\mathbf{arb}(\mathsf{F}_{|\{\mathsf{X}\}})^{[2]}$$

DEF maps$_4$: [Map range, i.e. second components of pairs in map] $\mathbf{range}(\mathsf{F}) =_{\mathrm{Def}}$

$$\{\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{F}\}$$

DEF maps$_5$: [Map predicate] $\mathsf{Is\_map}(\mathsf{F}) \leftrightarrow_{\mathrm{Def}}$

$$\left\langle \forall \mathsf{p} \in \mathsf{F} \mid \mathsf{p} = [\mathsf{p}^{[1]},\mathsf{p}^{[2]}] \right\rangle$$

DEF maps$_6$: [Single-valued map] $\mathsf{Svm}(\mathsf{F}) \leftrightarrow_{\mathrm{Def}}$

$$\mathsf{Is\_map}(\mathsf{F}) \;\&\; \left\langle \forall \mathsf{p} \in \mathsf{F}, \mathsf{q} \in \mathsf{F} \mid \mathsf{p}^{[1]} = \mathsf{q}^{[1]} \rightarrow \mathsf{p} = \mathsf{q} \right\rangle$$

DEF Finite : [Finitude] $\mathsf{Finite}(\mathsf{F}) \leftrightarrow_{\mathrm{Def}}$

$$\left\langle \forall \mathsf{g} \in \mathscr{P}(\mathscr{P}\mathsf{F}) \setminus \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g} \cap \mathscr{P}\mathsf{m} = \{\mathsf{m}\} \right\rangle$$

DEF HerFin: [Hereditary finitude] $\mathsf{HerFin}(\mathsf{S}) \leftrightarrow_{\mathrm{Def}}$

$$\mathsf{Finite}(\mathsf{S}) \;\&\; \left\langle \forall \mathsf{x} \in \mathsf{S} \mid \mathsf{HerFin}(\mathsf{x}) \right\rangle$$

Figure 1: Four properties refer to digraphs, all others to generic sets

mapping ('Svm')[3] and finitude, and the *recursive* property of hereditary finitude. This figure already shows the salient role of set abstraction terms—called, simply, *setformers*—in Ref: e.g., the setformer $\left\{\{\mathsf{p}^{[1]},\mathsf{p}^{[2]}\} : \mathsf{p} \in \mathsf{D} \mid \mathsf{p} = [\mathsf{p}^{[1]},\mathsf{p}^{[2]}]\right\}$ designates the set of all doubletons (or singletons) which result from the ordered pairs in D when the positions of their components are purposely forgotten.

The first definition in Fig. 1 specifies the property of a digraph (V,D) in which every non-null set w of vertices has a *sink*, namely a $\mathsf{t} \in \mathsf{w}$ devoid of outgoing edges $[\mathsf{t},\mathsf{y}]$ with $\mathsf{y} \in \mathsf{w}$; if there are finitely many edges, this amounts to forbidding cycles $[x_0,x_1],[x_1,x_2],\ldots,[x_k,x_0]$ of edges. The semantics of Ref's built-in **arb** operator, which picks from every set $w \neq \emptyset$ a $t = \mathbf{arb}(w)$ such that $t \in w$ and $\emptyset = \{y \in w \mid y \in t\}$ is analogous: in fact **arb** is meant to witness that membership is a well-founded relation, thus excluding cycles $x_0 \in x_1 \in \cdots \in x_k \in x_0$. (For definiteness, one also puts $\mathbf{arb}(\emptyset) = \emptyset$).

Fig. 2 collects various claims about the notions defined in Fig. 1: theorems (here reported without proofs), which will surface again in this paper.

Fig. 3 shows the formal development, with Ref, of a proof. Each one of the nine lines forming this proof, duly indicates which inference rule is employed to get the corresponding statement. This proof invokes twice a THEORY named finiteImage, whose interface is displayed in Fig. 4. While finiteImage does not return any symbol, the other, subtler THEORY displayed in the same figure, namely finiteInduction, returns a symbol, fin_Θ, representing an $\subseteq$-minimal set which meets P—given that at least one finite set satisfying property P exists. Likewise, the THEORY finiteAcycLabeling displayed in Fig. 5 returns a labeling of a given acyclic digraph, thereby furnishing the technique for decorating the graph *à la* Mostowski (see further on, upper part of Fig. 10).

Note that certain THEORYs, e.g. the one shown in Fig. 6, are encompassed by specialized inference

[3]To enforce a useful distinction, we denote by $G(x)$ the application of a *global* function G to an argument x ('global' meaning that the domain of G consists of all sets), while denoting by $f\upharpoonright x$ the application to x of a *map* f (typically single-valued), viewed as a set of pairs.

THM restr_2: [Each pair in a map belongs to the shoot of an element of its domain]
$\mathsf{P} = [\mathsf{X},\mathsf{Y}] \to (\mathsf{P} \in \mathsf{F}_{|\{\mathsf{Z}\}} \leftrightarrow (\mathsf{Z} = \mathsf{X} \;\&\; [\mathsf{Z},\mathsf{Y}] \in \mathsf{F}))$

THM image_4: [Meaning of application] $\mathsf{Svm}(\mathsf{F}) \;\&\; \mathsf{P} \in \mathsf{F} \to \mathsf{P} = [\mathsf{P}^{[1]}, \mathsf{F}\restriction\mathsf{P}^{[1]}]$

THM image_5: [Form of a single-valued map] $\mathsf{Svm}(\mathsf{F}) \leftrightarrow \mathsf{F} = \{[\mathsf{x}, \mathsf{F}\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{F})\}$

THM $\mathsf{singletonMap}_3$: [Transplant of singleton sub-map]
$\mathsf{Svm}(\mathsf{F}) \;\&\; [\mathsf{X},\mathsf{Y}] \in \mathsf{F} \;\&\; \mathsf{Z} \notin \mathbf{dom}(\mathsf{F}) \;\&\; \mathsf{G} = \mathsf{F} \setminus \{[\mathsf{X},\mathsf{Y}]\} \cup \{[\mathsf{Z},\mathsf{Y}]\} \to$
$\mathsf{Svm}(\mathsf{G}) \;\&\; \mathbf{dom}(\mathsf{G}) = \mathbf{dom}(\mathsf{F}) \setminus \{\mathsf{X}\} \cup \{\mathsf{Z}\} \;\&\; \mathbf{range}(\mathsf{G}) = \mathbf{range}(\mathsf{F})$

THM $\mathsf{vertexInduced}_0$: [Loop-freeness gets inherited] $\mathsf{E} \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{V}, \mathsf{y} \in \mathsf{V} \setminus \{\mathsf{x}\}\} \to$
$\mathsf{E} \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{W}, \mathsf{y} \in \mathsf{W}\} = \mathsf{E} \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{W}, \mathsf{y} \in \mathsf{W} \setminus \{\mathsf{x}\}\}$

THM $\mathsf{orientation}_0$: [Orientation does not take pseudo-edges into account] $\mathsf{W} \supseteq \mathsf{V} \to$
$(\mathsf{Orientates}(\mathsf{D},\mathsf{V},\mathsf{E}) \leftrightarrow \mathsf{Orientates}(\mathsf{D},\mathsf{V},\mathsf{E} \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{W}, \mathsf{y} \in \mathsf{W}\}))$

THM $\mathsf{voidgraph}_1$: [The void has all virtues]
$\mathsf{V} \subseteq \{\mathsf{X}\} \to \mathsf{Extensional}(\mathsf{V}, \emptyset) \;\&\; \mathsf{Orientates}(\emptyset, \mathsf{V}, \mathsf{E})$

THM $\mathsf{voidgraph}_2$: [The void has all virtues, contd.] $\mathsf{Acyclic}(\mathsf{V}, \emptyset)$

Figure 2: Miscellaneous theorems proved with Ref. The respective proofs consist of 11, 7, 24, 20, 5, 7, 9, and 6 inference lines

THM $\mathsf{part\_whole}_0$. $\mathsf{Svm}(\mathsf{F}) \to (\mathsf{Finite}(\mathsf{F}) \leftrightarrow \mathsf{Finite}(\mathbf{dom}(\mathsf{F})))$. PROOF:
$\mathsf{Suppose\_not}(\mathsf{f}_1) \Rightarrow$ AUTO
$\mathsf{Suppose} \Rightarrow \mathsf{Finite}(\mathsf{f}_1)$
$\mathsf{APPLY}\ \langle\,\rangle\ \mathsf{finiteImage}(\mathsf{s}_0 \mapsto \mathsf{f}_1, \mathsf{f}(\mathsf{X}) \mapsto \mathsf{X}^{[1]}) \Rightarrow \mathsf{Finite}(\{\mathsf{x}^{[1]} : \mathsf{x} \in \mathsf{f}_1\})$
$\mathsf{Use\_def}(\mathbf{dom}) \Rightarrow \mathsf{false}$
$\mathsf{Discharge} \Rightarrow$ AUTO
$\langle \mathsf{f}_1 \rangle \hookrightarrow T\mathsf{image}_5 \Rightarrow \mathsf{f}_1 = \{[\mathsf{x}, \mathsf{f}_1\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f}_1)\}$
$\mathsf{APPLY}\ \langle\,\rangle\ \mathsf{finiteImage}(\mathsf{s}_0 \mapsto \mathbf{dom}(\mathsf{f}_1), \mathsf{f}(\mathsf{X}) \mapsto [\mathsf{x}, \mathsf{f}_1\restriction\mathsf{x}]) \Rightarrow$
$\mathsf{Finite}(\{[\mathsf{x}, \mathsf{f}_1\restriction\mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f}_1)\})$
$\mathsf{EQUAL} \Rightarrow \mathsf{false}$
$\mathsf{Discharge} \Rightarrow$ QED

Figure 3: Example of a theorem proved in the Ref language

THEORY $\mathsf{finiteImage}(\mathsf{s}_0, \mathsf{f}(\mathsf{X}))$
$\mathsf{Finite}(\mathsf{s}_0)$
$\Rightarrow$
$\mathsf{Finite}(\{\mathsf{f}(\mathsf{x}) : \mathsf{x} \in \mathsf{s}_0\})$
END finiteImage

THEORY $\mathsf{finiteInduction}(\mathsf{s}_0, \mathsf{P}(\mathsf{S}))$
$\mathsf{Finite}(\mathsf{s}_0) \;\&\; \mathsf{P}(\mathsf{s}_0)$
$\Rightarrow (\mathsf{fin}_\ominus)$
$\langle \forall \mathsf{S} \mid \mathsf{S} \subsetneq \mathsf{fin}_\ominus \to \mathsf{Finite}(\mathsf{S}) \;\&\; (\mathsf{P}(\mathsf{S}) \leftrightarrow \mathsf{S} = \mathsf{fin}_\ominus) \rangle$
END finiteInduction

Figure 4: Interfaces of two THEORYs regarding finitude

THEORY $\mathsf{finAcycLabeling}(\mathsf{v}_0, \mathsf{d}_0, \mathsf{h}(\mathsf{S},\mathsf{X}))$
$\mathsf{Acyclic}(\mathsf{v}_0, \mathsf{d}_0) \;\&\; \mathsf{Finite}(\mathsf{v}_0)$
$\Rightarrow (\mathsf{lab}_\ominus)$
$\mathsf{Svm}(\mathsf{lab}_\ominus) \;\&\; \mathbf{dom}(\mathsf{lab}_\ominus) = \mathsf{v}_0$
$\langle \forall \mathsf{x} \in \mathsf{v}_0 \mid \mathsf{lab}_\ominus\restriction\mathsf{x} = \mathsf{h}(\{\mathsf{lab}_\ominus\restriction\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}\}} \mid \mathsf{p}^{[2]} \in \mathsf{v}_0\}, \mathsf{x}) \rangle$
END finAcycLabeling

Figure 5: Interface of a THEORY usable to label an acyclic digraph

rules eliminating the need to invoke them directly. In contrast, a built-in first-order Skolemization mechanism available in Ref has a close kinship to THEORYs, and hence gets invoked like them by means of the keyword APPLY . For example, inside the THEORY finiteInduction of which Fig. 4 shows the interface, fin_Θ gets formalized in two steps: one proves

$$\text{THM } \mathsf{finiteInduction}_0.\ \big\langle \exists m \mid \{s \subseteq s_0 \mid P(s)\} \cap \mathscr{P} m = \{m\}\big\rangle$$

in the first place and then, by invoking

$$\text{APPLY } \big\langle v_{1\Theta} : \mathsf{fin}_\Theta \big\rangle \text{ Skolem} \Rightarrow \text{THM } \mathsf{finiteInduction}_1.\ \{s \subseteq s_0 \mid P(s)\} \cap \mathscr{P}\mathsf{fin}_\Theta = \{\mathsf{fin}_\Theta\}\,,$$

assigns a name to an entity satisfying the existential claim of that theorem. Likewise, after having constructed the THEORY finiteInduction, by invoking

$$\begin{array}{l} \text{APPLY } \big\langle \mathsf{fin}_\Theta : w_1 \big\rangle \ \mathsf{finiteInduction}\big(s_0 \mapsto w_2, \\ \qquad P(W) \mapsto (W \subseteq v_0 \ \& \ W \neq \emptyset \ \& \ \neg\big\langle \exists t \in W \mid \emptyset = \{y \in W \mid [y,t] \in d_0\}\big\rangle)\big) \Rightarrow \\ \big\langle \forall v \mid v \subseteq w_1 \rightarrow \mathsf{Finite}(v) \ \& \\ \qquad (v \subseteq v_0 \ \& \ v \neq \emptyset \ \& \ \neg\big\langle \exists t \in v \mid \emptyset = \{y \in v \mid [y,t] \in d_0\}\big\rangle \leftrightarrow v = w_1)\big\rangle\,, \end{array}$$

we instantiate the constant w_1 as needed inside the proof shown in Fig. 7.

$$\begin{array}{l} \text{THEORY } \mathsf{isSvm}\big(s_0, f(X), P(X)\big) \\ \Rightarrow \\ \qquad \mathsf{Svm}\big(\{[x, f(x)] : x \in s_0 \mid P(x)\}\big) \\ \qquad \mathbf{dom}(\{[x, f(x)] : x \in s_0 \mid P(x)\}) = \{x \in s_0 \mid P(x)\} \ \& \ \{x \in s_0 \mid \mathsf{true}\} = s_0 \\ \qquad \mathbf{range}(\{[x, f(x)] : x \in s_0 \mid P(x)\}) = \{f(x) : x \in s_0 \mid P(x)\} \\ \text{END } \mathsf{isSvm} \end{array}$$

Figure 6: Interface of a THEORY recognizing a mapping, of which it shows domain and range

$$\begin{array}{l} \text{THM } \mathsf{acyclicity}_3\text{: [Local sources in an acyclic graph]} \\ \quad W \neq \emptyset \ \& \ \mathsf{Acyclic}(V, D) \ \& \ \mathsf{Finite}(V) \ \& \ V \supseteq W \rightarrow \\ \qquad\qquad \big\langle \exists t \in W \mid \emptyset = \{y \in W \mid [y,t] \in D\}\big\rangle. \text{ PROOF:} \\ \mathsf{Suppose\_not}(w_2, v_0, d_0) \Rightarrow \ \text{AUTO} \\ \quad \mathsf{Use\_def}(\mathsf{Acyclic}) \Rightarrow \ \mathit{Stat1}: \ \big\langle \forall w \subseteq v_0 \mid w \neq \emptyset \rightarrow \big\langle \exists t \in w \mid \emptyset = \{y \in w \mid [t,y] \in d_0\}\big\rangle\big\rangle \ \& \\ \qquad\qquad \mathsf{Finite}(w_2) \\ \quad \text{APPLY } \big\langle \mathsf{fin}_\Theta : w_1 \big\rangle \ \mathsf{finiteInduction}\big(s_0 \mapsto w_2, P(W) \mapsto \\ \qquad\qquad (W \subseteq v_0 \ \& \ W \neq \emptyset \ \& \ \neg\big\langle \exists t \in W \mid \emptyset = \{y \in W \mid [y,t] \in d_0\}\big\rangle)\big) \Rightarrow \ \mathit{Stat7}: \\ \qquad\qquad \big\langle \forall v \mid v \subseteq w_1 \rightarrow \mathsf{Finite}(v) \ \& \\ \qquad\qquad\qquad (v \subseteq v_0 \ \& \ v \neq \emptyset \ \& \ \neg\big\langle \exists t \in v \mid \emptyset = \{y \in v \mid [y,t] \in d_0\}\big\rangle \leftrightarrow v = w_1)\big\rangle \\ \quad \big\langle w_1 \big\rangle \hookrightarrow \mathit{Stat7} \Rightarrow \ \mathit{Stat8}: \ \neg\big\langle \exists t \in w_1 \mid \emptyset = \{y \in w_1 \mid [y,t] \in d_0\}\big\rangle \ \& \ w_1 \neq \emptyset \ \& \ w_1 \subseteq v_0 \\ \quad \big\langle w_1 \big\rangle \hookrightarrow \mathit{Stat1} \Rightarrow \ \mathit{Stat9}: \ \big\langle \exists t \in w_1 \mid \emptyset = \{y \in w_1 \mid [t,y] \in d_0\}\big\rangle \\ \quad \big\langle a \big\rangle \hookrightarrow \mathit{Stat9} \Rightarrow \ \mathit{Stat10}: \ \{y \in w_1 \mid [a,y] \in d_0\} = \emptyset \ \& \ a \in w_1 \\ \quad \mathsf{Suppose} \Rightarrow \ w_1 = \{a\} \\ \qquad \big\langle a \big\rangle \hookrightarrow \mathit{Stat8} \Rightarrow \ \mathit{Stat11}: \ \{y \in w_1 \mid [y,a] \in d_0\} \neq \emptyset \\ \qquad \big\langle c \big\rangle \hookrightarrow \mathit{Stat11} \Rightarrow \ [a,a] \in d_0 \\ \qquad \big\langle a \big\rangle \hookrightarrow \mathit{Stat10} \Rightarrow \ \mathsf{false} \\ \quad \mathsf{Discharge} \Rightarrow \ \text{AUTO} \\ \quad \big\langle w_1 \setminus \{a\} \big\rangle \hookrightarrow \mathit{Stat7} \Rightarrow \ \mathit{Stat14}: \ \big\langle \exists t \in w_1 \setminus \{a\} \mid \emptyset = \{y \in w_1 \setminus \{a\} \mid [y,t] \in d_0\}\big\rangle \\ \quad \big\langle t_0 \big\rangle \hookrightarrow \mathit{Stat14} \Rightarrow \ \{y \in w_1 \setminus \{a\} \mid [y,t_0] \in d_0\} = \emptyset \ \& \ t_0 \in w_1 \setminus \{a\} \\ \quad \big\langle t_0 \big\rangle \hookrightarrow \mathit{Stat8} \Rightarrow \ \mathit{Stat16}: \ \{y \in w_1 \mid [y,t_0] \in d_0\} \neq \emptyset \\ \quad \big\langle b \big\rangle \hookrightarrow \mathit{Stat16} \Rightarrow \ \mathit{Stat17}: \ b \notin \{y \in w_1 \setminus \{a\} \mid [y,t_0] \in d_0\} \ \& \ [b,t_0] \in d_0 \ \& \ b \in w_1 \\ \quad \big\langle b \big\rangle \hookrightarrow \mathit{Stat17} \Rightarrow \ [a,t_0] \in d_0 \\ \quad \big\langle t_0 \big\rangle \hookrightarrow \mathit{Stat10} \Rightarrow \ \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \ \text{QED} \end{array}$$

Figure 7: Another example of a theorem proved in the Ref language

Proving the claim of the theorem in Fig. 7 amounts, in practice, to showing that the 'converse' of the acyclic digraph under consideration, namely the digraph where the direction of each edge has been inverted, is also acyclic: straightforward changes to this proof, with only two additional inference lines, would in fact lead to a proof of

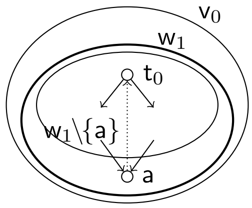

Figure 8: A graphical representation of the proof of THM $\mathsf{acyclicity}_3$ from Fig. 7

$$\mathsf{Acyclic}(\mathsf{V},\mathsf{D})\ \&\ \mathsf{Finite}(\mathsf{V}) \rightarrow \mathsf{Acyclic}(\mathsf{V},\left\{\left[\mathsf{p}^{[2]},\mathsf{p}^{[1]}\right] : \mathsf{p}\in\mathsf{D}\,|\,\mathsf{p}=\left[\mathsf{p}^{[1]},\mathsf{p}^{[2]}\right]\right\}).$$

The eighteen lines shown in Fig. 7 form an argument by contradiction, which goes as follows (cf. Fig. 8). Suppose that the acyclic digraph $(\mathsf{v}_0,\mathsf{d}_0)$ and a non-null set w_2 of its vertices make a counterexample to the claim; use finite induction to get a *minimal* w_1 constituting in its turn, in combination with $(\mathsf{v}_0,\mathsf{d}_0)$, a counter-example. Hence, $\mathsf{v}_0 \supseteq \mathsf{w}_1$, $\mathsf{w}_1 \neq \emptyset$, and each $\mathsf{t} \in \mathsf{w}_1$ has at least one entering edge $[\mathsf{y},\mathsf{t}] \in \mathsf{d}_0$ with $\mathsf{y} \in \mathsf{w}_1$. On the other hand, acyclicity ensures us that there is an $\mathsf{a} \in \mathsf{w}_1$ devoid of outgoing edges $[\mathsf{a},\mathsf{y}]$ with $\mathsf{y} \in \mathsf{w}_1$; which implies $\mathsf{w}_1 \neq \{\mathsf{a}\}$, else we would readily get the contradiction $[\mathsf{a},\mathsf{a}] \in \mathsf{d}_0\ \&\ [\mathsf{a},\mathsf{a}] \notin \mathsf{d}_0$. Thanks to the minimality of w_1, we know there is a $\mathsf{t}_0 \in \mathsf{w}_1 \setminus \{\mathsf{a}\}$ devoid of entering edges $[\mathsf{a},\mathsf{y}]$ with $\mathsf{y} \in \mathsf{w}_1 \setminus \{\mathsf{a}\}$; but then $[\mathsf{a},\mathsf{t}_0]$ must be the edge entering t_0 in w_1, which leads us to the sought contradiction. As the reader will perceive at once from Fig. 7, the formal counterpart of this argument resorts extensively to substitutions of new constants for existential variables and of suitably chosen terms for universal variables; but notice: these classical inference mechanisms are enabled, in Ref, to also interact with setformers.

As illustrated by Fig. 1, it is often expedient to formulate definitions in rather liberal terms. For example, the short comments associated with the definitions maps_1 through maps_4 suggest that the argument F of the **dom**, restriction, application, and **range** operations is typically a *map*, viz. a set of pairs; but it would be pedantry to constrain in this sense the formal definitions, causing more complicated theorem statements and, consequently, unduly cumbersome proofs. Likewise, the notions of acyclicity and extensionality are usually referred to a pair (V, D) where D (representing the set of edges of a digraph) is included in the Cartesian square of the set V of vertices, which in its turn is typically finite. But we feel no need to enforce this: for, it proves at times useful, in inductive arguments about graphs (cf. the proof in Fig. 7), to consider a smaller and smaller subset of an initial set of vertices without bothering to narrow the set of edges correspondingly. Then it will go without saying that we are considering vertex-induced subgraphs of the initial graph.

Still in Fig. 1, when it comes to specifying an orientation D of a graph (V, E), we neither insist that E must be included in the set $\{\,\{\mathsf{x},\mathsf{y}\} : \mathsf{x}\in\mathsf{V},\mathsf{y}\in\mathsf{V}\setminus\{\mathsf{x}\}\,\}$ of doubletons, nor that D must consist of pairs. More simply, we choose to ignore the part of E which is not formed by the said doubletons and the part of D which is not a map.

Occasionally it pays off to extend this liberal attitude to the higher THEORY level: in Fig. 5, for example, we are not putting the condition $\mathsf{d}_0 \subseteq \mathsf{v}_0 \times \mathsf{v}_0$ among the assumptions of finAcycLabeling, even though it will be met in typical applications of this THEORY . However, the type-free set-theoretic foundation of Ref will not prevent us from choosing, in specific situations, a less liberal attitude. Cautiousness will emerge later on, as shown by the assumptions of the THEORYs in Fig. 10; and the finitude assumption of the just cited THEORY finAcycLabeling already offers an instance of it. That assumption, in fact, while reflecting a customary way of looking at graphs, goes against a habit of set-theorists, who primarily speculate about infinite entities (cf. [2]).

Before ending this section, we want to stress that it is often unnecessary to package a group of theorems into an autonomous THEORY : recourse to a THEORY is appropriate when (as in the cases described in Figures 4 and 5) either the proofs of a group of statements depend on some common global function or predicate, or there is a rationale for concealing the details of the definition of some new symbol. When the cohesion of a group of theorems only lies in the fact that they concern the same notions, cf. e.g. Figures 9 and 16, then proving them consecutively inside the same *scenario* (viz. in the same proof-script file) should be enough to ensure their convenient usability.

THM acyclicity$_0$: [Adjunction of an outer vertex to a digraph cannot disrupt acyclicity]
$\mathsf{V} \times \mathsf{V} \supseteq \mathsf{D} \,\&\, \mathsf{X} \notin \mathsf{V} \,\&\, \mathsf{V} \supseteq \mathsf{S} \,\&\, \mathsf{Acyclic}(\mathsf{V},\mathsf{D}) \rightarrow \mathsf{Acyclic}\big(\mathsf{V} \cup \{\mathsf{X}\}, \mathsf{D} \cup (\{\mathsf{X}\} \times \mathsf{S})\big)$

THM acyclicity$_1$: [Reduction of the set of edges of a digraph preserves its acyclicity]
$\mathsf{Acyclic}(\mathsf{V},\mathsf{D}) \,\&\, \mathsf{V}' \subseteq \mathsf{V} \,\&\, \mathsf{D}' \subseteq \mathsf{D} \rightarrow \mathsf{Acyclic}(\mathsf{V}',\mathsf{D}')$

THM acyclicity$_2$: [Acyclic digraphs are devoid of self-loops and of symmetrical arcs]
$\mathsf{Acyclic}(\mathsf{V},\mathsf{D}) \,\&\, \{\mathsf{Y},\mathsf{X}\} \subseteq \mathsf{V} \,\&\, [\mathsf{X},\mathsf{Y}] \in \mathsf{D} \rightarrow [\mathsf{Y},\mathsf{X}] \notin \mathsf{D} \,\&\, \mathsf{X} \neq \mathsf{Y}$

THM acyclicity$_4$: [Every acyclic graph has sinks and sources]
$\mathsf{Acyclic}(\mathsf{V},\mathsf{D}) \,\&\, \mathsf{Finite}(\mathsf{V}) \,\&\, \mathsf{V} \neq \emptyset \rightarrow$
$\big\langle \exists \mathsf{s} \in \mathsf{V}, \mathsf{t} \in \mathsf{V} \mid \emptyset = \{\mathsf{y} \in \mathsf{V} \mid [\mathsf{s},\mathsf{y}] \in \mathsf{D} \vee [\mathsf{y},\mathsf{t}] \in \mathsf{D}\} \big\rangle$

THM acyclicity$_5$: [No triangle inside an acyclic digraph]
$\mathsf{Acyclic}(\mathsf{V},\mathsf{D}) \,\&\, \{\mathsf{X},\mathsf{Y},\mathsf{Z}\} \subseteq \mathsf{V} \,\&\, \{[\mathsf{X},\mathsf{Y}],[\mathsf{Y},\mathsf{Z}]\} \subseteq \mathsf{D} \rightarrow [\mathsf{Z},\mathsf{X}] \notin \mathsf{D}$

THM acyclicity$_6$: [Adjunction of an inner vertex to a digraph cannot disrupt acyclicity]
$\mathsf{V} \times \mathsf{V} \supseteq \mathsf{D} \,\&\, \mathsf{X} \notin \mathsf{V} \,\&\, \mathsf{V} \supseteq \mathsf{S} \,\&\, \mathsf{Acyclic}(\mathsf{V},\mathsf{D}) \rightarrow \mathsf{Acyclic}\big(\mathsf{V} \cup \{\mathsf{X}\}, \mathsf{D} \cup (\mathsf{S} \times \{\mathsf{X}\})\big)$

Figure 9: Properties enjoyed by acyclicity

THEORY finMostowskiDecoration$(\mathsf{v}_0, \mathsf{d}_0)$
$\mathsf{v}_0 \times \mathsf{v}_0 \supseteq \mathsf{d}_0 \,\&\, \mathsf{v}_0 \neq \emptyset \,\&\, \mathsf{Finite}(\mathsf{v}_0) \,\&\, \mathsf{Acyclic}(\mathsf{v}_0,\mathsf{d}_0) \,\&\, \mathsf{WExtensional}(\mathsf{v}_0,\mathsf{d}_0)$
$\Rightarrow$ (mski$_\ominus$)
$\mathsf{Svm}(\mathsf{mski}_\ominus) \,\&\, \mathbf{dom}(\mathsf{mski}_\ominus) = \mathsf{v}_0$
$\big\langle \forall \mathsf{w} \mid \mathsf{w} \in \mathbf{dom}(\mathsf{d}_0) \longrightarrow \mathsf{mski}_\ominus \restriction \mathsf{w} = \{\mathsf{mski}_\ominus \restriction \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0 \restriction \{\mathsf{w}\}}\} \,\&\, \mathsf{mski}_\ominus \restriction \mathsf{w} \neq \emptyset \big\rangle$
$\emptyset \in \mathbf{range}(\mathsf{mski}_\ominus) \,\&\, \big\langle \forall \mathsf{y} \mid \mathsf{y} \in \mathbf{range}(\mathsf{mski}_\ominus) \longrightarrow \mathsf{Finite}(\mathsf{y}) \big\rangle$
$\big\langle \forall \mathsf{x},\mathsf{y} \mid \{\mathsf{x},\mathsf{y}\} \subseteq \mathsf{v}_0 \,\&\, \mathsf{mski}_\ominus \restriction \mathsf{x} = \mathsf{mski}_\ominus \restriction \mathsf{y} \longrightarrow \mathsf{x} = \mathsf{y} \big\rangle$
$\big\langle \forall \mathsf{y} \mid \mathsf{y} \in \mathsf{v}_0 \longrightarrow (\mathsf{mski}_\ominus \restriction \mathsf{y} \in \mathsf{mski}_\ominus \restriction \mathsf{x} \leftrightarrow [\mathsf{x},\mathsf{y}] \in \mathsf{d}_0) \big\rangle$
END finMostowskiDecoration

THEORY finGraphRepr$(\mathsf{v}_0, \mathsf{e}_0)$
$\mathsf{e}_0 \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0 \setminus \{\mathsf{x}\}\} \,\&\, \mathsf{v}_0 \neq \emptyset \,\&\, \mathsf{Finite}(\mathsf{v}_0)$
$\Rightarrow$ (wski$_\ominus$)
$\mathsf{Svm}(\mathsf{wski}_\ominus) \,\&\, \mathbf{dom}(\mathsf{wski}_\ominus) = \mathsf{v}_0 \,\&\, \emptyset \in \mathbf{range}(\mathsf{wski}_\ominus)$
$\big\langle \forall \mathsf{y} \mid \mathsf{y} \in \mathbf{range}(\mathsf{wski}_\ominus) \longrightarrow \mathsf{Finite}(\mathsf{y}) \big\rangle$
$\big\langle \forall \mathsf{x},\mathsf{y} \mid \{\mathsf{x},\mathsf{y}\} \subseteq \mathsf{v}_0 \,\&\, \mathsf{wski}_\ominus \restriction \mathsf{x} = \mathsf{wski}_\ominus \restriction \mathsf{y} \longrightarrow \mathsf{x} = \mathsf{y} \big\rangle$
$\big\langle \forall \mathsf{x},\mathsf{y} \mid \{\mathsf{x},\mathsf{y}\} \subseteq \mathsf{v}_0 \longrightarrow ((\mathsf{wski}_\ominus \restriction \mathsf{y} \in \mathsf{wski}_\ominus \restriction \mathsf{x} \vee \mathsf{wski}_\ominus \restriction \mathsf{x} \in \mathsf{wski}_\ominus \restriction \mathsf{y}) \leftrightarrow \{\mathsf{x},\mathsf{y}\} \in \mathsf{e}_0) \big\rangle$
$\big\langle \forall \mathsf{x} \mid \mathsf{wski}_\ominus \restriction \mathsf{x} \cap \mathbf{range}(\mathsf{wski}_\ominus) \neq \emptyset \longrightarrow \mathsf{wski}_\ominus \restriction \mathsf{x} \subseteq \mathbf{range}(\mathsf{wski}_\ominus) \big\rangle$
END finGraphRepr

Figure 10: THEORYs on Mostowski's decoration and on graph representation

3 Basic edge-to-membership translation

As said in the introduction, our first experiment amounts to showing how:

(1) to convert an arbitrary undirected graph into a weakly extensional acyclic digraph,

(2) to decorate the digraph resulting from (1) by sets, so that its edges mirror membership.

This overall formalization task, and its subtask (2), culminate in the two THEORYs shown in Fig. 10. In particular, the THEORY finMostowskiDecoration implements (2); while the key theorem, corresponding to (1), which makes the THEORY finGraphRepr easily obtainable from the other one is stated in Ref as follows:

THM xtensionalization$_0$. $\mathsf{Finite}(\mathsf{V}) \,\&\, \mathsf{S} \in \mathsf{V} \rightarrow$
$\big\langle \exists \mathsf{d} \mid \mathsf{Orientates}(\mathsf{d},\mathsf{V},\mathsf{E}) \,\&\, \mathsf{Acyclic}(\mathsf{V},\mathsf{d}) \,\&\, \mathsf{WExtensional}(\mathsf{V},\mathsf{d}) \,\&\, \mathsf{S} \notin \mathbf{range}(\mathsf{d}) \big\rangle$.

In view of its centrality in our scenario, we wish to briefly sketch the proof of the orientability theorem xtensionalization$_0$ cited above, whose specification in Ref required 71 proof lines. Arguing by contradiction, suppose that there is a counterexample; then, exploiting the finiteness hypothesis, take a *minimal* counterexample $\mathsf{v}_1, \mathsf{s}_1, \mathsf{e}_0$. We are supposing that there is no acyclic, weakly extensional orientation of the graph $\big(\mathsf{v}_1, \mathsf{e}_0 \cap \big\{\{x,y\} : x \in \mathsf{v}_1,\, y \in \mathsf{v}_1 \setminus \{x\}\big\}\big)$ having s_1 as a source; whereas, for every $\mathsf{v}_0 \subsetneq \mathsf{v}_1$, one can orient $\big(\mathsf{v}_0, \mathsf{e}_0 \cap \big\{\{x,y\} : x \in \mathsf{v}_0,\, y \in \mathsf{v}_0 \setminus \{x\}\big\}\big)$ by an acyclic and weakly extensional

$\mathsf{d}_0 \subseteq \mathsf{v}_0 \times \mathsf{v}_0$, for any vertex $t \in \mathsf{v}_0$, so that t plays the role of a source. Let, in particular, $\mathsf{v}_0 = \mathsf{v}_1 \setminus \{\mathsf{s}_1\}$. Unless s_1 is an isolated vertex, an acyclic and weakly extensional orientation of v_0 exists that has as a source a chosen neighbor t_1 of s_1 (see Fig. 11). However, that orientation could trivially be extended into a weakly extensional acyclic orientation of the graph with vertices v_1 so that s_1 becomes a source; this contradiction shows that s_1 cannot have neighbors in v_1, which is also untenable: any orientation for v_0, in fact, works also as an orientation for v_1 and, as such, has each isolated vertex of v_1—in particular s_1—as a source.

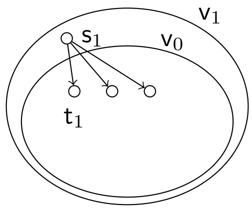

Figure 11: Extending a weakly extensional acyclic orientation of $\big(\mathsf{v}_0, \mathsf{e}_0 \cap \big\{\{x,y\} : x \in \mathsf{v}_0,\ y \in \mathsf{v}_0 \setminus \{x\}\big\}\big)$ in which a neighbor t_1 of s_1 acts as a source, by making s_1 a source

Alongside with the issue of getting a weakly extensional orientation of any given graph, another important issue is how to construct, inside the THEORY finMostowskiDecoration, the labeling mski_Θ of the given acyclic, weakly extensional digraph $(\mathsf{v}_0, \mathsf{d}_0)$ so that it meets the desired conditions (e.g., injectivity over v_0). To clarify what will follow, let us see how to express certain properties that a vertex s can enjoy in *any* digraph $(\mathsf{v}_0\,,\,\mathsf{d}_0)$, i.e. simply under assumption that $\mathsf{d}_0 \subseteq \mathsf{v}_0 \times \mathsf{v}_0$:

$$\begin{array}{ll} s \text{ is a source:} & s \in \mathsf{v}_0 \setminus \mathbf{range}(\mathsf{d}_0)\,, \\ s \text{ is a sink:} & s \in \mathsf{v}_0 \setminus \mathbf{dom}(\mathsf{d}_0)\,, \\ s \text{ is an out-neighbor of } x\text{:} & s \in \big\{\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{x\}}\big\}\,(= \mathbf{range}(\mathsf{d}_{0|\{x\}}))\,. \end{array}$$

The construction of mski_Θ can be carried out in many ways, and we have opted for the following rather simple technique (see Fig. 12). Consider the global functions

$$\begin{array}{lcl} \mathsf{lbl}(\mathsf{W}) & = & \left\{ \begin{array}{ll} \textbf{if} & \mathsf{W} \in \mathbf{dom}(\mathsf{d}_0) \cup \{\mathbf{arb}(\mathsf{v}_0 \backslash \mathbf{dom}(\mathsf{d}_0))\} \\ \textbf{then} & \emptyset \\ \textbf{else} & \{\{\mathsf{v}_0\} \cup (\mathsf{v}_0 \backslash \{\mathsf{W}\})\} \quad \textbf{fi}\,, \end{array} \right. \\ \mathsf{h}(\mathsf{S}, \mathsf{X}) & = & \mathsf{S} \cup \mathsf{lbl}(\mathsf{X})\,, \end{array}$$

and use this h to instantiate the third parameter of finAcycLabeling. Putting $\mathsf{mski}_\Theta = \mathsf{lab}_\Theta$ will automatically enforce the condition (stated with a slight redundancy)

$$\mathsf{mski}_\Theta{\upharpoonright}\mathsf{x} \quad = \quad \big\{\mathsf{mski}_\Theta{\upharpoonright}\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}\}} \mid \mathsf{p}^{[2]} \in \mathsf{v}_0\big\} \cup \mathsf{lbl}(\mathsf{x})$$

for all $\mathsf{x} \in \mathsf{v}_0$; moreover, mski_Θ readily turns out to be a function sending each vertex to a finite set and sending the sink $\mathbf{arb}(\mathsf{v}_0 \backslash \mathbf{dom}(\mathsf{d}_0))$ to $\emptyset$.

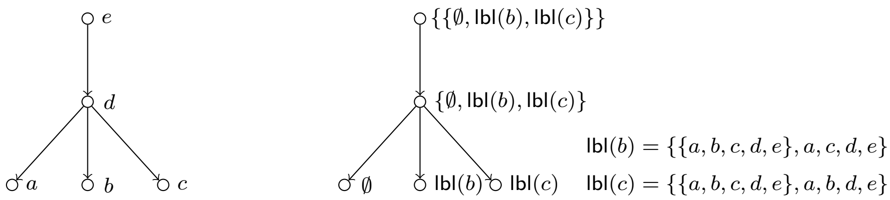

Figure 12: Injective labeling of a weakly extensional acyclic digraph with vertex set $v_0 = \{a,b,c,d,e\}$

The injectivity of mski_Θ is obvious over the set $\mathsf{v}_0 \setminus \mathbf{dom}(\mathsf{d}_0)$ of all sinks (because, there, lbl is injective and mski_Θ and lbl take the same values); from the sinks it easily extends to all other vertices: *cardinality considerations* (see below) show in fact that a value-collision between a sink and an internal vertex is impossible, and the weak extensionality assumption prevents collisions to occur between internal vertices. Still inside finMostowskiDecoration, one derives from the injectivity of mski_Θ that $u \in \mathsf{mski}_\Theta{\upharpoonright}\mathsf{x}$ is satisfied if and only if either $u = \mathsf{mski}_\Theta\lceil \mathsf{y}$ and $[\mathsf{x},\mathsf{y}] \in \mathsf{d}_0$, or u is the set—if any—satisfying $\{u\} = \mathsf{lbl}(\mathsf{x})$. Note that the second case of this alternative vanishes when the digraph has just one sink, i.e. it is extensional.

Let us take a closer look at the 'cardinality considerations' which we have just alluded to. For the purposes of the Ref scenario on which we are reporting, we do not need a theory of cardinals of any sophistication: instead, since we mostly deal with finite sets, we can rely on various facts on finitude such as

THM fin_0. $\mathsf{Y} \supseteq \mathsf{X} \,\&\, \mathsf{Finite}(\mathsf{Y}) \rightarrow \mathsf{Finite}(\mathsf{X})$,
THM fin_1. $\mathsf{Finite}(\mathsf{F}) \rightarrow \mathsf{Finite}(\mathsf{F} \cup \{\mathsf{X}\})$,
THM fin_2. $\mathsf{Finite}(\{\mathsf{X}\}) \,\&\, \mathsf{Finite}(\emptyset)$,
THM $\mathsf{part\_whole}_0$. $\mathsf{Svm}(\mathsf{F}) \rightarrow \big(\mathsf{Finite}(\mathsf{F}) \leftrightarrow \mathsf{Finite}(\mathbf{dom}(\mathsf{F}))\big)$,
THM $\mathsf{part\_whole}_1$. $\mathsf{Svm}(\mathsf{H}) \,\&\, \mathsf{Finite}(\mathsf{H}) \,\&\, \mathbf{range}(\mathsf{H}) \supseteq \mathbf{dom}(\mathsf{H}) \rightarrow \mathbf{range}(\mathsf{H}) = \mathbf{dom}(\mathsf{H})$,

the last of which (which costs us just a 25-line Ref proof) states that 'the part is smaller than the whole'.

These—especially THM $\mathsf{part\_whole}_1$—, as we will now see, enable suppression of any explicit reference to the notion of cardinality from the proof of

THM $\mathsf{finMostowskiDecoration}_7$. $\{\mathsf{X},\mathsf{Y}\} \subseteq \mathsf{v}_0 \,\&\, \mathsf{mski}_\Theta\lceil\mathsf{Y} \in \mathsf{mski}_\Theta\lceil\mathsf{X} \rightarrow \mathsf{X} \in \mathbf{dom}(\mathsf{d}_0)$,

whose claim means that the label of a vertex never belongs to the label of a sink. This statement, inside the THEORY finMostowskiDecoration, shall be exploited in its turn to prove the important injectivity and isomorphism claims

THM $\mathsf{finMostowskiDecoration}_9$. $\{\mathsf{X},\mathsf{Y}\} \subseteq \mathsf{v}_0 \,\&\, \mathsf{mski}_\Theta\lceil\mathsf{X} = \mathsf{mski}_\Theta\lceil\mathsf{Y} \rightarrow \mathsf{X} = \mathsf{Y}$,
THM $\mathsf{finMostowskiDecoration}_{10}$. $\mathsf{Y} \in \mathsf{v}_0 \rightarrow (\mathsf{mski}_\Theta\lceil\mathsf{Y} \in \mathsf{mski}_\Theta\lceil\mathsf{X} \leftrightarrow [\mathsf{X},\mathsf{Y}] \in \mathsf{d}_0)$.

The proof of THM $\mathsf{finMostowskiDecoration}_7$ roughly goes as follows. Arguing by contradiction, assume that $\mathsf{mski}_\Theta\lceil\mathsf{w}_1 \in \mathsf{mski}_\Theta\lceil\mathsf{w}_0$, where $\mathsf{w}_0, \mathsf{w}_1$ are a sink and a vertex of $(\mathsf{v}_0, \mathsf{d}_0)$. Since w_0 is a sink, we have $\mathsf{mski}_\Theta\lceil\mathsf{w}_0 = \mathsf{lbl}(\mathsf{w}_0) = \{\{\mathsf{v}_0\} \cup (\mathsf{v}_0 \backslash \{\mathsf{w}_0\})\}$; therefore $\mathsf{mski}_\Theta\lceil\mathsf{w}_1 = \{\mathsf{v}_0\} \cup (\mathsf{v}_0 \backslash \{\mathsf{w}_0\})$, and w_1 cannot be a sink. Thus $\mathsf{mski}_\Theta\lceil\mathsf{w}_1 = \big\{\mathsf{mski}_\Theta\lceil\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{w}_1\}}\big\}$. A contradiction, here, lies in the fact that $\{\mathsf{v}_0\} \cup (\mathsf{v}_0 \backslash \{\mathsf{w}_0\})$ has a cardinality—the same as v_0—which $\big\{\mathsf{mski}_\Theta\lceil\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{w}_1\}}\big\}$ cannot reach: this is because $\mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})$—where $\mathsf{p}^{[2]}$ takes its values—is a strict subset of v_0, since $\mathsf{Acyclic}(\mathsf{v}_0, \mathsf{d}_0)$ implies $\mathsf{w}_1 \in \mathsf{v}_0 \setminus \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})$, by THM $\mathsf{acyclicity}_2$ (cf. Fig. 9).

Within our Ref scenario we have managed to formalize the ending of the above argument-by-contradiction in the following slicker (albeit slightly less intuitive) terms, taking advantage of THM $\mathsf{part\_whole}_1$ to avoid talking about cardinalities.

Put $\mathsf{h}_0 = \big\{[\mathsf{y}, \mathsf{mski}_\Theta\lceil\mathsf{y}] : \mathsf{y} \in \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})\big\}$, so that h_0 is single-valued, $\mathbf{dom}(\mathsf{h}_0) = \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}}) \subseteq \mathsf{v}_0$, $\mathbf{range}(\mathsf{h}_0) = \big\{\mathsf{mski}_\Theta\lceil\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{w}_1\}}\big\} = \{\mathsf{v}_0\} \cup (\mathsf{v}_0 \backslash \{\mathsf{w}_0\})$, and hence $\mathsf{v}_0 \in \mathbf{range}(\mathsf{h}_0)$.

Momentarily suppose that $\mathsf{w}_0 \notin \mathbf{dom}(\mathsf{h}_0)$; then $\mathbf{dom}(\mathsf{h}_0) \subseteq \mathbf{range}(\mathsf{h}_0)$ and we can resort to THM $\mathsf{part\_whole}_1$ to get $\mathbf{range}(\mathsf{h}_0) = \mathbf{dom}(\mathsf{h}_0)$; but then $\mathsf{v}_0 \in \mathbf{range}(\mathsf{h}_0) = \mathbf{dom}(\mathsf{h}_0) \subseteq \mathsf{v}_0$ must hold, whence a contradiction readily arises, because $v \notin v$ holds for any v. On the other hand, if $\mathsf{w}_0 \in \mathbf{dom}(\mathsf{h}_0)$ then we can retouch h_0 by replacing its pair $[\mathsf{w}_0, \mathsf{h}_0\lceil\mathsf{w}_0]$ by $[\mathsf{w}_1, \mathsf{h}_0\lceil\mathsf{w}_0]$. Since $\mathsf{w}_1 \notin \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})$, we get in this manner a single-valued map h_1 which has the same range as the original h_0 and is still finite. It will satisfy $\mathbf{dom}(\mathsf{h}_1) \subseteq \mathbf{range}(\mathsf{h}_1)$, enabling derivation of $\mathsf{v}_0 \in \mathbf{range}(\mathsf{h}_1) = \mathbf{dom}(\mathsf{h}_1) \subseteq \mathsf{v}_0$ via THM $\mathsf{part\_whole}_1$, hence leading us to a contradiction again.

Fig. 13 shows the formal counterpart of the proof just outlined. Note that the keyword PROOF which normally precedes the beginning of a Ref's proof is here boosted by a suffixed '+' sign, which activates the behind-the-scenes type-inference mechanism discussed in [11, pp. 122-127] and in [12, Section 4.3.7]. That mechanism exempts us from having to explicitly cite any of the THMs fin_0, fin_1, and $\mathsf{part\_whole}_0$ cited above. Similarly, the inference rule TELEM hides exploitation of the THEORY isSvm seen in Fig. 6, while SIMPLF has a certain ability to unravel setformers, and Set_monot to detect inclusions between them. Besides the already mentioned $\mathsf{part\_whole}_1$ and some of the THMs displayed in Fig. 2, the proof under discussion cites the following:

THM $\mathsf{finMostowskiDecoration}_1$: [No self-loops in an acyclic digraph] $\mathsf{W} \notin \mathbf{range}(\mathsf{d}_{0|\{\mathsf{W}\}})$,
THM $\mathsf{finMostowskiDecoration}_3$: [Images of internal vertices under Mostowski's decoration]
$\mathsf{W} \in \mathbf{dom}(\mathsf{d}_0) \rightarrow \mathsf{mski}_\Theta\lceil\mathsf{W} = \Big\{\mathsf{mski}_\Theta\lceil\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{W}\}}\Big\} \,\&\, \mathsf{mski}_\Theta\lceil\mathsf{W} \neq \emptyset$,
THM $\mathsf{finMostowskiDecoration}_4$: [Images of sinks under Mostowski's decoration]
$\mathsf{W} \in \mathsf{v}_0 \backslash \mathbf{dom}(\mathsf{d}_0) \rightarrow \mathsf{mski}_\Theta\lceil\mathsf{W} = \mathsf{lbl}(\mathsf{W}) \,\&\, \mathsf{mski}_\Theta\lceil\mathsf{W} \notin \{\{\mathsf{v}_0\} \cup (\mathsf{v}_0 \backslash \{\emptyset\})\}$.

THM finMostowskiDecoration$_7$. $\{\mathsf{X},\mathsf{Y}\} \subseteq \mathsf{v}_0 \,\&\, \mathsf{mski}_\ominus\lceil\mathsf{Y} \in \mathsf{mski}_\ominus\lceil\mathsf{X} \rightarrow \mathsf{X} \in \mathbf{dom}(\mathsf{d}_0)$. PROOF+:

Suppose_not$(\mathsf{w}_0,\mathsf{w}_1) \Rightarrow$ AUTO

Use_def$(\mathsf{lbl}(\mathsf{w}_0)) \Rightarrow$ AUTO

$\langle \mathsf{w}_0\rangle \hookrightarrow T$finMostowskiDecoration$_4(\star) \Rightarrow \mathsf{w}_1 \in \mathsf{v}_0 \,\&\, \mathsf{mski}_\ominus\lceil\mathsf{w}_1 = \{\mathsf{v}_0\} \cup (\mathsf{v}_0 \setminus \{\mathsf{w}_0\})$

Suppose $\Rightarrow \mathsf{w}_1 \notin \mathbf{dom}(\mathsf{d}_0)$

$\langle \mathsf{w}_1\rangle \hookrightarrow T$finMostowskiDecoration$_4 \Rightarrow \mathsf{lbl}(\mathsf{w}_1) = \{\mathsf{v}_0\} \cup (\mathsf{v}_0 \setminus \{\mathsf{w}_0\})$

Use_def$(\mathsf{lbl}(\mathsf{w}_1)) \Rightarrow$ AUTO

Discharge $\Rightarrow$ AUTO

$\langle \mathsf{w}_1\rangle \hookrightarrow T$finMostowskiDecoration$_3 \Rightarrow$

$\{\mathsf{mski}_\ominus\lceil\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{w}_1\}}\} = \{\mathsf{v}_0\} \cup (\mathsf{v}_0 \setminus \{\mathsf{w}_0\}) \,\&\, \mathsf{w}_1 \in \mathbf{dom}(\mathsf{d}_0)$

Loc_def $\Rightarrow \mathsf{h}_0 = \{[\mathsf{y},\mathsf{mski}_\ominus\lceil\mathsf{y}] : \mathsf{y} \in \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})\}$

TELEM $\Rightarrow \mathsf{Svm}(\{[\mathsf{y},\mathsf{mski}_\ominus\lceil\mathsf{y}] : \mathsf{y} \in \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})\}) \,\&$
$\mathbf{range}(\{[\mathsf{y},\mathsf{mski}_\ominus\lceil\mathsf{y}] : \mathsf{y} \in \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})\}) = \{\mathsf{mski}_\ominus\lceil\mathsf{y} : \mathsf{y} \in \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})\} \,\&$
$\mathbf{dom}(\{[\mathsf{y},\mathsf{mski}_\ominus\lceil\mathsf{y}] : \mathsf{y} \in \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})\}) = \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})$

Use_def$(\mathbf{range}) \Rightarrow \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}}) = \{\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{w}_1\}}\}$

SIMPLF $\Rightarrow \{\mathsf{mski}_\ominus\lceil\mathsf{y} : \mathsf{y} \in \{\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{w}_1\}}\}\} = \{\mathsf{mski}_\ominus\lceil\mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{w}_1\}}\}$

Assump $\Rightarrow \mathsf{Finite}(\mathsf{v}_0) \,\&\, \mathsf{Acyclic}(\mathsf{v}_0,\mathsf{d}_0) \,\&\, \mathsf{v}_0 \neq \emptyset \,\&\, \mathsf{v}_0 \times \mathsf{v}_0 \supseteq \mathsf{d}_0$

Suppose $\Rightarrow \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}}) \not\subseteq \mathsf{v}_0$

Use_def$(|) \Rightarrow \{\mathsf{p}^{[2]} : \mathsf{p} \in \{\mathsf{q} \in \mathsf{d}_0 \mid \mathsf{q}^{[1]} \in \{\mathsf{w}_1\}\}\} \not\subseteq \mathsf{v}_0$

Set_monot $\Rightarrow \{q^{[2]} : \mathsf{q} \in \mathsf{d}_0 \mid \mathsf{q}^{[1]} \in \{\mathsf{w}_1\}\} \subseteq \{q^{[2]} : \mathsf{q} \in \mathsf{d}_0 \mid \mathsf{q}^{[1]} \in \mathsf{v}_0 \times \mathsf{v}_0\}$

Use_def$(\times) \Rightarrow \{q^{[2]} : \mathsf{q} \in \mathsf{d}_0 \mid \mathsf{q}^{[1]} \in \{\mathsf{w}_1\}\} \subseteq \{q^{[2]} : \mathsf{q} \in \{[\mathsf{x},\mathsf{y}] : \mathsf{x} \in \mathsf{v}_0,\, \mathsf{y} \in \mathsf{v}_0\}\}$

SIMPLF $\Rightarrow$ *Stat1* : $\left\{[\mathsf{x},\mathsf{y}]^{[2]} : \mathsf{x} \in \mathsf{v}_0,\, \mathsf{y} \in \mathsf{v}_0\right\} \not\subseteq \mathsf{v}_0$

$\langle \mathsf{y}_1\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2* : $\mathsf{y}_1 \in \left\{[\mathsf{x},\mathsf{y}]^{[2]} : \mathsf{x} \in \mathsf{v}_0,\, \mathsf{y} \in \mathsf{v}_0\right\} \,\&\, \mathsf{y}_1 \notin \mathsf{v}_0$

$\langle \mathsf{x}',\mathsf{y}'\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false

Discharge $\Rightarrow$ *Stat3* : $\mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}}) \subseteq \mathsf{v}_0$

EQUAL $\Rightarrow \mathsf{Svm}(\mathsf{h}_0) \,\&\, \mathbf{range}(\mathsf{h}_0) = \{\mathsf{v}_0\} \cup (\mathsf{v}_0 \setminus \{\mathsf{w}_0\}) \,\&\, \mathbf{dom}(\mathsf{h}_0) \subseteq \mathsf{v}_0 \,\&\, \mathsf{Finite}(\mathsf{h}_0)$

Use_def$(\mathbf{dom}(\mathsf{h}_0)) \Rightarrow$ AUTO

$\langle \mathsf{h}_0\rangle \hookrightarrow T$part_whole$_1 \Rightarrow$ *Stat6* : $\mathsf{w}_0 \in \{\mathsf{p}^{[1]} : \mathsf{p} \in \mathsf{h}_0\}$

$\langle \mathsf{p}_0\rangle \hookrightarrow$ *Stat6* $\Rightarrow \mathsf{p}_0 \in \mathsf{h}_0 \,\&\, \mathsf{w}_0 = \mathsf{p}_0^{[1]}$

$\langle \mathsf{h}_0,\mathsf{p}_0\rangle \hookrightarrow T$image$_4 \Rightarrow \mathsf{p}_0 = [\mathsf{p}_0^{[1]}, \mathsf{h}_0\lceil\mathsf{p}_0^{[1]}]$

Loc_def $\Rightarrow$ *Stat7* : $\mathsf{y}_0 = \mathsf{h}_0\lceil\mathsf{p}_0^{[1]} \,\&\, \mathsf{h}_1 = \mathsf{h}_0 \setminus \{[\mathsf{w}_0,\mathsf{y}_0]\} \cup \{[\mathsf{w}_1,\mathsf{y}_0]\}$

EQUAL $\Rightarrow [\mathsf{w}_0,\mathsf{y}_0] \in \mathsf{h}_0 \,\&\, \mathsf{Finite}(\mathsf{h}_1) \,\&\, \mathbf{dom}(\mathsf{h}_0) = \mathbf{range}(\mathsf{d}_{0|\{\mathsf{w}_1\}})$

$\langle \mathsf{w}_1\rangle \hookrightarrow T$finMostowskiDecoration$_1 \Rightarrow \mathsf{w}_1 \in \mathsf{v}_0 \setminus \{\mathsf{w}_0\} \setminus \mathbf{dom}(\mathsf{h}_0)$

$\langle \mathsf{h}_0,\mathsf{w}_0,\mathsf{y}_0,\mathsf{w}_1,\mathsf{h}_1\rangle \hookrightarrow T$singletonMap$_3$(*Stat3*$\star$) $\Rightarrow$
$\mathsf{Svm}(\mathsf{h}_1) \,\&\, \mathbf{dom}(\mathsf{h}_1) \subseteq \mathsf{v}_0 \setminus \{\mathsf{w}_0\} \,\&\, \mathbf{range}(\mathsf{h}_1) = \{\mathsf{v}_0\} \cup (\mathsf{v}_0 \setminus \{\mathsf{w}_0\})$

$\langle \mathsf{h}_1\rangle \hookrightarrow T$part_whole$_1$(*Stat7*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Figure 13: Formalized proof of a major theorem needed to prove our decoration injective

4 Representing connected claw-free graphs as membership digraphs

Our richer construction must associate with each connected claw-free graph $G = (V, E)$ an injection f from V onto a transitive, hereditarily finite set ν_G so that $\{x, y\} \in E$ if and only if either $f\,x \in f\,y$ or $f\,y \in f\,x$.

The new notions entering into play are rendered formally as follows:

$$\begin{aligned}
\mathsf{ClawFreeG}(\mathsf{V},\mathsf{E}) \quad &\leftrightarrow_{\mathrm{Def}} \quad \langle \forall \mathsf{w},\mathsf{x},\mathsf{y},\mathsf{z} \mid \{\mathsf{w},\mathsf{x},\mathsf{y},\mathsf{z}\} \subseteq \mathsf{V} \,\&\, \{\{\mathsf{w},\mathsf{y}\},\{\mathsf{y},\mathsf{x}\},\{\mathsf{y},\mathsf{z}\}\} \subseteq \mathsf{E} \rightarrow \\
&\qquad (\mathsf{x} = \mathsf{z} \vee \mathsf{w} \in \{\mathsf{z},\mathsf{x}\} \vee \{\mathsf{x},\mathsf{z}\} \in \mathsf{E} \vee \{\mathsf{z},\mathsf{w}\} \in \mathsf{E} \vee \{\mathsf{w},\mathsf{x}\} \in \mathsf{E})\rangle\,, \\
\mathsf{Connected}(\mathsf{E}) \quad &\leftrightarrow_{\mathrm{Def}} \quad \mathsf{E} \not\subseteq \{\emptyset\} \wedge \{\mathsf{b} \subseteq \mathsf{E} \mid \bigcup \mathsf{b} \cap \bigcup(\mathsf{E} \setminus \mathsf{b}) = \emptyset\} \subseteq \{\emptyset, \mathsf{E}\}\,.
\end{aligned}$$

Here, the first *definiens* requires that no subgraph of (V, E) induced by four vertices has the shape of a 'Y' (see Fig. 14). The second one requires that the set E of edges can nohow be partitioned into multiple vertex-disjoint blocks.

A fact that we will need is that every connected graph has a vertex whose removal (along with all edges incident to it) does not disrupt connectivity; for example, each white vertex in Fig. 15 enjoys this property. The existence of such a NON-CUT VERTEX is proved with relative ease for a tree—nevertheless the proof of this fact, as formulated in THM tree$_1$ of Fig. 16, turned out to be the longest in our scenario. So, in order to cheaply achieve our goal, we define[4]

[4]Concerning the notion of hank, generalizing the notion of cycle, cf. [10, Sec. 4.4].

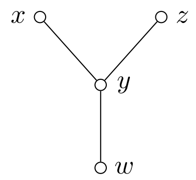

Figure 14: The claw $K_{1,3}$

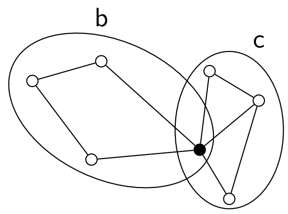

Figure 15: A partition of the set of edges of a connected graph; b and c, the blocks of this partition, are not vertex-disjoint

$$\begin{array}{rcl} \mathsf{HankFree}(\mathsf{T}) & \leftrightarrow_{\mathrm{Def}} & \left\langle \forall \mathsf{e} \subseteq \mathsf{T} \mid \mathsf{e} = \emptyset \vee \left\langle \exists\, \mathsf{a} \in \mathsf{e} \mid \mathsf{a} \not\subseteq \bigcup(\mathsf{e} \setminus \{\mathsf{a}\}) \right\rangle \right\rangle, \\ \mathsf{Is\_tree}(\mathsf{T}) & \leftrightarrow_{\mathrm{Def}} & \mathsf{Connected}(\mathsf{T}) \ \wedge \ \mathsf{HankFree}(\mathsf{T}), \end{array}$$

and recast, to then use it in the THEORY shown in Fig. 17, the connectivity assumption as the equivalent one that $(\mathsf{v}_0, \mathsf{e}_0)$ has a 'spanning tree':

$$\mathsf{HasSpanningTree}(\mathsf{V},\ \mathsf{E}) \quad \leftrightarrow_{\mathrm{Def}} \quad \left\langle \exists \mathsf{t} \mid \mathsf{Is\_tree}(\mathsf{t}) \,\&\, \bigcup \mathsf{t} = \mathsf{V} \,\&\, (\mathsf{V} = \{\mathbf{arb}(\mathsf{V})\} \vee \mathsf{t} \subseteq \mathsf{E}) \right\rangle.$$

This eases things: for, any vertex with fewer than 2 incident edges in the spanning tree of a connected graph easily turns out to be a non-cut vertex of the graph, as summarized by THM $\mathsf{connectivity}_2$ of Fig. 16.

THM tree_0: [A tree cannot be null or have a null edge] $\mathsf{Is\_tree}(\mathsf{T}) \rightarrow \emptyset \notin \mathsf{T} \cup \{\mathsf{T}\}$

THM tree_1: [Non-singleton trees can be pruned]
$$\mathsf{Is\_tree}(\mathsf{T}) \,\&\, \mathsf{T} \neq \{\mathbf{arb}(\mathsf{T})\} \,\&\, \mathsf{T} \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{p} \in \mathsf{T}, \mathsf{x} \in \mathsf{p}, \mathsf{y} \in \mathsf{p}\} \rightarrow \left\langle \exists \mathsf{e} \in \mathsf{T}, \mathsf{u} \in \mathsf{e} \mid \{\mathsf{a} \in \mathsf{T} \mid \mathsf{u} \notin \mathsf{a}\} = \mathsf{T} \setminus \{\mathsf{e}\} \,\&\, \mathsf{Is\_tree}(\mathsf{T} \setminus \{\mathsf{e}\}) \right\rangle$$

THM tree_2: [Every singleton other than $\{\emptyset\}$ is a tree] $\mathsf{A} \neq \emptyset \leftrightarrow \mathsf{Is\_tree}(\{\mathsf{A}\})$

THM tree_4: [In a tree obtained by removing an edge from a tree, only one vertex gets lost]
$$\mathsf{Is\_tree}(\mathsf{T}) \,\&\, \{\mathsf{X},\mathsf{Y}\} = \mathsf{A} \,\&\, \mathsf{A} \in \mathsf{T} \,\&\, \mathsf{Is\_tree}(\mathsf{T} \setminus \{\mathsf{A}\}) \,\&\, \{\mathsf{e} \in \mathsf{T} \mid \mathsf{X} \notin \mathsf{e}\} = \mathsf{T} \setminus \{\mathsf{A}\} \rightarrow \bigcup(\mathsf{T} \setminus \{\mathsf{A}\}) = \bigcup \mathsf{T} \setminus \{\mathsf{X}\}$$

THM $\mathsf{connectivity}_1$: [No vertex is isolated in a graph endowed with a spanning tree]
$$\mathsf{HasSpanningTree}(\mathsf{V},\mathsf{E}) \,\&\, \mathsf{E} \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{Z}, \mathsf{y} \in \mathsf{Z} \setminus \{\mathsf{x}\}\} \,\&\, \mathsf{U} \in \mathsf{V} \,\&\, \mathsf{V} \setminus \{\mathsf{U}\} \neq \emptyset \rightarrow \left\langle \exists \mathsf{w} \in \mathsf{V} \setminus \{\mathsf{U}\} \mid \{\mathsf{U},\mathsf{w}\} \in \mathsf{E} \right\rangle$$

THM $\mathsf{connectivity}_2$: [Every graph endowed with a spanning tree has a non-cut vertex]
$$\mathsf{HasSpanningTree}(\mathsf{V},\mathsf{E}) \,\&\, \mathsf{E} \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{V}, \mathsf{y} \in \mathsf{V} \setminus \{\mathsf{x}\}\} \,\&\, \mathsf{V} \neq \{\mathbf{arb}(\mathsf{V})\} \rightarrow \left\langle \exists \mathsf{u} \in \mathsf{V} \mid \mathsf{HasSpanningTree}(\mathsf{V} \setminus \{\mathsf{u}\}, \{\mathsf{a} \in \mathsf{E} \mid \mathsf{u} \notin \mathsf{a}\}) \right\rangle$$

Figure 16: Properties enjoyed by trees and, respectively, by graphs endowed with spanning trees. THM tree_1, implying that trees have non-cut vertices, called for a 90-line Ref proof

We now aim at getting the analogue, shown in Fig. 17, of the THEORY finGraphRepr discussed in Section 3 (cf. Fig. 10). For that, we must again exploit the THEORY finMostowskiDecoration; in addition, a key theorem will ensure the acyclic extensional orientability of a connected and claw-free graph:

THM $\mathsf{cClawFreeG}_2$.
$$\mathsf{Finite}(\mathsf{V}) \,\&\, \mathsf{HasSpanningTree}(\mathsf{V},\mathsf{E}) \,\&\, \mathsf{ClawFreeG}(\mathsf{V},\mathsf{E}) \,\&\, \mathsf{E} \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{V}, \mathsf{y} \in \mathsf{V} \setminus \{\mathsf{x}\}\} \rightarrow \left\langle \exists \mathsf{d} \subseteq \mathsf{V} \times \mathsf{V} \mid \mathsf{Orientates}(\mathsf{d},\mathsf{V},\mathsf{E}) \,\&\, \mathsf{Acyclic}(\mathsf{V},\mathsf{d}) \,\&\, \mathsf{Extensional}(\mathsf{V},\mathsf{d}) \right\rangle.$$

$\textsc{Theory}$ herfinCCFGraphRepr(v_0, e_0)

$\quad e_0 \subseteq \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$ & $\mathsf{Finite}(v_0)$

$\quad \mathsf{HasSpanningTree}(v_0, e_0)$ & $\mathsf{ClawFreeG}(v_0, e_0)$

$\Rightarrow$ (trans_Θ)

$\quad \mathsf{Svm}(\mathsf{trans}_\Theta)$ & $\mathbf{dom}(\mathsf{trans}_\Theta) = v_0$

$\quad \langle \forall x, y \mid \{X, Y\} \subseteq v_0 \mathbin{\&} \mathsf{trans}_\Theta \upharpoonright X = \mathsf{trans}_\Theta \upharpoonright Y \rightarrow X = Y \rangle$

$\quad \langle \forall x, y \mid \{X, Y\} \subseteq v_0 \rightarrow$

$\qquad\qquad (\mathsf{trans}_\Theta \upharpoonright Y \in \mathsf{trans}_\Theta \upharpoonright X \vee \mathsf{trans}_\Theta \upharpoonright X \in \mathsf{trans}_\Theta \upharpoonright Y \leftrightarrow \{X, Y\} \in e_0) \rangle$

$\quad \{y \in \mathbf{range}(\mathsf{trans}_\Theta) \mid y \not\subseteq \mathbf{range}(\mathsf{trans}_\Theta)\} = \emptyset$

$\quad \mathbf{range}(\mathsf{trans}_\Theta) \neq \emptyset$ & $\mathsf{HerFin}(\mathbf{range}(\mathsf{trans}_\Theta))$

$\textsc{End}$ herfinCCFGraphRepr

Figure 17: $\textsc{Theory}$ on representing a connected claw-free graph via membership

Let us outline the proof of this $\textsc{Thm}$, whose Ref formalization is shown in appendix. Arguing by contradiction, we assume that there is a counterexample (v_2, e_2) to the claim. Then, thanks to the finiteness hypothesis, we can take a minimal counterexample (v_1, e_1) with $v_1 \subseteq v_2$ and $e_1 = e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}$. Note that v_1 cannot be a singleton, else a contradiction would arise: the null set of edges would in fact be an extensional, acyclic orientation of (v_1, e_1).

Since v_1 is not a singleton we can, thanks to $\textsc{Thm}$ $\mathsf{connectivity}_2$ of Fig. 16, consider a non-cut vertex x_0 of (v_1, e_1). Now consider the graph (v_0, e_0) induced by (v_1, e_1) on the strict subset $v_1 \setminus \{x_0\}$ of the set of vertices. This graph inherits the claw-freeness property, due to the easy

$\textsc{Thm}$ $\mathsf{cClawFreeG}_0$. $\mathsf{ClawFreeG}(V, E) \mathbin{\&} W \subseteq V \rightarrow \mathsf{ClawFreeG}(W, \{a \in E \mid a \subseteq W\})$;

therefore, the minimality assumption concerning v_1 ensures us that we can obtain an extensional acyclic orientation d_0 of this induced graph.

We first deal with the case when the sink of the acyclic digraph $(v_1 \setminus \{x_0\}, d_0)$ it not adjacent to x_0 through e_1 (see Fig. 18, left). In this case, as suggested by the $\textsc{Thm}$ $\mathsf{cClawFreeG}_1$ shown below (which has a Ref proof of 73 lines), we orient the edges incident to x_0 as out-going from x_0, to get an extensional acyclic orientation d_1 for (v_1, e_1). Note that the neighbors of x_0 through e_1 are $\{t \in v_1 \mid \{x_0, t\} \in e_1\}$, and hence $d_1 = d_0 \cup (\{x_0\} \times \{t \in v_1 \mid \{x_0, t\} \in e_1\})$.

Let us briefly digress to give clues about the proof of the auxiliary lemma

$\textsc{Thm}$ $\mathsf{cClawFreeG}_1$. $W = V \cup \{U\} \mathbin{\&} U \notin V \mathbin{\&} \{s \in V \mid D_{|\{s\}} = \emptyset \mathbin{\&} \{s, U\} \in E\} = \emptyset \mathbin{\&}$

$\qquad E \subseteq \{\{x,y\} : x \in Z, y \in Z \setminus \{x\}\} \mathbin{\&} \mathsf{ClawFreeG}(W, E) \mathbin{\&} \mathsf{HasSpanningTree}(W, E) \mathbin{\&}$

$\qquad \mathsf{Orientates}(D, V, E) \mathbin{\&} \mathsf{Acyclic}(V, D) \mathbin{\&} \mathsf{Extensional}(V, D) \mathbin{\&} D \subseteq V \times V \mathbin{\&}$

$\qquad D' = D \cup (\{U\} \times \{t \in V \mid \{U, t\} \in E\}) \rightarrow$

$\qquad\quad \mathsf{Orientates}(D', W, E) \mathbin{\&} \mathsf{Acyclic}(W, D') \mathbin{\&} \mathsf{Extensional}(W, D') \mathbin{\&} D' \subseteq W \times W$

as instantiated for the purposes of the case at hand, namely with $W = v_1$, $V = v_1 \setminus \{x_0\}$, $U = x_0$, $D = d_0$, $E = e_1$, $Z = v_2$, and $D' = d_1$. Consider the digraph in which we orient all edges incident to x_0 as out-going from x_0; this is acyclic, by $\textsc{Thm}$ $\mathsf{acyclicity}_0$ (cf. Fig. 9). Assume for a contradiction that there exists an $x_1 \in v_1 \setminus \{x_0\}$ having the same out-neighborhood as x_0. Since, by $\textsc{Thm}$ $\mathsf{connectivity}_1$, x_0 is not an isolated vertex, the set $\{t \in v_1 \mid \{x_0, t\} \in e_1\}$ of neighbors of x_1 through e_1 is non-null. Since $\mathsf{Acyclic}(v_1 \setminus \{x_0\}, d_0)$ holds, we can consider a vertex $y_0 \in \{t \in v_1 \mid \{x_0, t\} \in e_1\}$ having no successors in common with x_0. Vertex y_0 is not a sink of $(v_1 \setminus \{x_0\}, d_0)$ by our initial assumption, thus there exists a successor z_0 of y_0 which is neither adjacent to x_0 nor to x_1, in consequence of the choice of y_0, of the fact that x_0 and x_1 have the same out-neighbors, and of $\textsc{Thm}$ $\mathsf{acyclicity}_5$. Since also x_0 and x_1 are not adjacent, by $\textsc{Thm}$ $\mathsf{acyclicity}_2$, it follows that the set $\{x_0, x_1, y_0, z_0\}$ is a claw of (v_1, e_2), a contradiction.

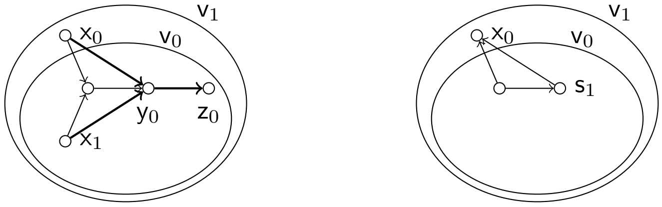

Figure 18: The main cases in the proof of $\textsc{Thm}$ $\mathsf{cClawFreeG}_2$. On the left, the acyclic digraph $(v_1 \setminus \{x_0\}, d_0)$ has no sink adjacent to x_0 through e_1; on the right, the complementary case

Next we deal with the case when the sink s_1 of $(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d}_0)$ is adjacent to x_0 through e_1 (see Fig. 18, right). Here we resort to the auxiliary lemma

$$\textsc{Thm}\ \mathsf{xtensionalization}_2.\ \mathsf{W} = \mathsf{V} \cup \{\mathsf{U}\}\ \&\ \mathsf{U} \notin \mathsf{V}\ \&\ \mathsf{S} \in \mathsf{V}\ \&\ \{\mathsf{y} \in \mathsf{V} \mid [\mathsf{S},\mathsf{y}] \in \mathsf{D}\} = \emptyset\ \&$$
$$\mathsf{S} \in \{\mathsf{t} \in \mathsf{W} \mid \{\mathsf{U},\mathsf{t}\} \in \mathsf{E}\}\ \&\ \mathsf{E} \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{Z}, \mathsf{y} \in \mathsf{Z} \setminus \{\mathsf{x}\}\}\ \&$$
$$\mathsf{Orientates}(\mathsf{D},\mathsf{V},\mathsf{E})\ \&\ \mathsf{Acyclic}(\mathsf{V},\mathsf{D})\ \&\ \mathsf{Extensional}(\mathsf{V},\mathsf{D})\ \&\ \mathsf{D} \subseteq \mathsf{V} \times \mathsf{V} \rightarrow$$
$$\big\langle \exists \mathsf{d} \subseteq \mathsf{W} \times \mathsf{W} \mid \mathsf{Orientates}(\mathsf{d},\mathsf{W},\mathsf{E})\ \&\ \mathsf{Acyclic}(\mathsf{W},\mathsf{d})\ \&\ \mathsf{Extensional}(\mathsf{W},\mathsf{d})\big\rangle$$

which, for our purposes, gets instantiated with $\mathsf{W} = \mathsf{v}_1$, $\mathsf{V} = \mathsf{v}_1 \setminus \{\mathsf{x}_0\}$, $\mathsf{U} = \mathsf{x}_0$, $\mathsf{S} = \mathsf{s}_1$, $\mathsf{D} = \mathsf{d}_0$, $\mathsf{E} = \mathsf{e}_2$, and $\mathsf{Z} = \mathsf{v}_2$.

The construction of d carried out inside the (51-line) proof of this Thm simply consists in orienting all edges incident to x_0 as in-coming to x_0: thus an acyclic d results, by Thm $\mathsf{acyclicity}_6$ (cf. Fig. 9), and d has x_0 as its unique sink; moreover, d is extensional because s_1 has x_0 as its sole out-neighbor, whereas every other vertex in $\mathsf{v}_1 \setminus \{\mathsf{x}_0\}$ has at least one other vertex in $\mathsf{v}_1 \setminus \{\mathsf{x}_0\}$ as out-neighbor.

Conclusions

The formalization experiment on which we have reported in Section 4 responds to a referee of our previous paper [10], who expressed the wish to see a Ref-checked proof of the representation theorem for connected claw-free graphs. We gladly accepted the challenge because, as we claimed in the introductory section of [10], it is precisely in the light of the said representation theorem that the change of perspective proposed there (with claw-free sets in place of claw-free graphs) acquires its full significance. The new Ref scenario hence is a due companion to our former one.

Thanks to plain definitions of various graph-theoretic definitions, e.g. acyclicity, we were able to implement most proofs without a big effort (the proofs of the six claims in Fig. 9, for example, required 30, 7, 7, 25, 15, and 27 inference lines). Yet, since the present bottleneck is the proof of Thm tree_1 (cf. Fig. 16), we feel obliged to deepen the formalization—which could have not belonged to this paper—of graph connectivity.

As reported in Section 3, we have also proved with Ref a representation result referring to a graph whatsoever, whose formal verification had been promised in [9]. This other result lies at a more fundamental level than the representation, through membership digraphs, of graphs belonging to special classes (connected claw-free graphs, graphs endowed with a Hamiltonian path, cf. [5]). Its experimental set up and the proof techniques involved are pretty much the same as for the other case study, but the intermediate acyclic digraph now turns out to be *weakly* extensional instead of just extensional; hence it would be modeled more naturally through a set with atoms than through one belonging to von Neumann's renowned *cumulative hierarchy* [14]. However, cf. [1, p. 54]:

> *Even in this case, one might still wish to prevent the existence of unrestricted atoms. In any case, for the "genuine" sets, Extensionality holds and the other sets are merely harmless curiosities.*

To get rid of such 'harmless curiosities' as atoms, we had to design a technique which, in the end, would remain hidden inside our Theory finMostowskiDecoration. Nevertheless we wanted our technique to be as light as possible, because sooner or later we will need similar techniques to handle more challenging situations, involving—as we expect—infinite graphs. The reader will judge whether we have achieved our goal, at least so far, parsimoniously enough.

Our representation theorems exploit sets demandingly: not only have we gone beyond the conventional view that the edges of a graph / digraph simply are doubletons / ordered pairs, but also, as just recalled, we have eliminated atoms from our sets. Also, we have required that the set representing a claw-free graph be transitive. Putting heavy restraints in the formulation of representation theorems is essential in order that a verifier well versed only about first principles can indeed serve as a proof assistant in specific domains.

Proof-verification can highly benefit from representation theorems of the kind illustrated in this paper. On the human side, such results disclose new insights by shedding light on a discipline from unusual angles; on the technological side, they enable the transfer of proof methods from one realm of mathematics to another. This opinion made us invest, in parallel with the studies reported above, in the celebrated theorem about representing Boolean algebras through Stone spaces. Reporting about graphs deserved priority, though, because we see issues regarding them as pre-algorithmic and, as such, application-oriented. Even the two propositions on the orientability of graphs discussed above are based on two algorithms of which, in a very definite sense, they prove the correctness.

Acknowledgements

We owe much to Salvatore Paxia, who helped us with a new release of the SETL2 programming language which supports the implementation of Ref. This enabled us to install a new, better performing, version of Ref.

Alberto Casagrande suggested the simplified definition of connectivity shown in Section 4 in place of the original less convenient one, which was:

$$\mathsf{Connected}(\mathsf{E}) \quad \leftrightarrow_{\mathrm{Def}} \quad \emptyset \notin \mathsf{E} \wedge$$
$$\left\langle \forall \mathsf{p} \mid \left(\bigcup \mathsf{p} = \mathsf{E} \wedge \left\langle \forall \mathsf{b} \in \mathsf{p}, \mathsf{c} \in \mathsf{p} \mid \bigcup \mathsf{b} \cap \bigcup \mathsf{c} \neq \emptyset \;\leftrightarrow\; \mathsf{b} = \mathsf{c} \right\rangle \right) \rightarrow \mathsf{p} = \{\mathsf{E}\} \right\rangle .$$

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A Ref proof on connected claw-free graphs

Acyclic extensional orientability of a connected, claw-free graph

THM cClawFreeG_2.
$\mathsf{Finite}(V)\ \&\ \mathsf{HasSpanningTree}(V,E)\ \&\ E \subseteq \{\{x,y\} : x \in V, y \in V \setminus \{x\}\}\ \&\ \mathsf{ClawFreeG}(V,E) \rightarrow$
$\langle \exists d \subseteq V \times V \mid \mathsf{Orientates}(d,V,E)\ \&\ \mathsf{Acyclic}(V,d)\ \&\ \mathsf{Extensional}(V,d)\rangle$. PROOF:

1 $\mathsf{Suppose\_not}(v_2, e_2) \Rightarrow$ AUTO

Arguing by contradiction, suppose that there is a counterexample v_2, e_2 to the claim. Then, thanks to the finiteness hypothesis, we can take a minimal counterexample v_1, e_1 with $v_1 \subseteq v_2$ and $e_1 = e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}$.

2 $\langle e_2, v_2, v_2\rangle \hookrightarrow T\mathsf{vertexInduced}_0 \Rightarrow$ $e_2 \cap \{\{x,y\} : x \in v_2, y \in v_2\} = e_2\ \&\ e_2 = e_2 \cap \{\{x,y\} : x \in v_2, y \in v_2 \setminus \{x\}\}$

3 EQUAL $\Rightarrow$ $\mathsf{HasSpanningTree}(v_2, e_2 \cap \{\{x,y\} : x \in v_2, y \in v_2\})\ \&$
$\mathsf{ClawFreeG}(v_2, e_2 \cap \{\{x,y\} : x \in v_2, y \in v_2\})\ \&$
$\neg\langle \exists d \subseteq v_2 \times v_2 \mid \mathsf{Orientates}(d, v_2, e_2)\ \&\ \mathsf{Acyclic}(v_2, d)\ \&\ \mathsf{Extensional}(v_2, d)\rangle$

4 APPLY $\langle \mathsf{fin}_\Theta : v_1\rangle$ $\mathsf{finiteInduction}\big(s_0 \mapsto v_2, P(S) \mapsto ($
$\mathsf{HasSpanningTree}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})\ \&$
$\mathsf{ClawFreeG}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})\ \&$
$\neg\langle \exists d \subseteq S \times S \mid \mathsf{Orientates}(d, S, e_2)\ \&\ \mathsf{Acyclic}(S, d)\ \&\ \mathsf{Extensional}(S, d)\rangle)\big) \Rightarrow$

$\mathit{Stat3}: \langle \forall S \mid S \subseteq v_1 \rightarrow \mathsf{Finite}(S)\ \&\ ($ $\mathsf{HasSpanningTree}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})\ \&$
$\mathsf{ClawFreeG}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})\ \&$
$\neg\langle \exists d \subseteq S \times S \mid \mathsf{Orientates}(d, S, e_2)\ \&\ \mathsf{Acyclic}(S, d)\ \&\ \mathsf{Extensional}(S, d)\rangle \leftrightarrow$ $S = v_1)\rangle$

5 $\langle v_1\rangle \hookrightarrow \mathit{Stat3} \Rightarrow$ $\mathsf{HasSpanningTree}(v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\})\ \&$
$\mathsf{ClawFreeG}(v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\})\ \&$
$\mathit{Stat4}: \neg\langle \exists d \subseteq v_1 \times v_1 \mid \mathsf{Orientates}(d, v_1, e_2)\ \&\ \mathsf{Acyclic}(v_1, d)\ \&\ \mathsf{Extensional}(v_1, d)\rangle$

We exclude that v_1 can be a singleton, else a contradiction would arise. In this case, in fact, an extensional acyclic orientation of v_1, e_1 is the null set of edges.

6 Suppose $\Rightarrow$ $v_1 = \{\mathbf{arb}(v_1)\}$

7 $\langle \emptyset\rangle \hookrightarrow \mathit{Stat4} \Rightarrow$ AUTO

8 $\langle v_1, \mathbf{arb}(v_1), e_2\rangle \hookrightarrow T\mathsf{voidgraph}_1 \Rightarrow$ AUTO

9 $\langle v_1\rangle \hookrightarrow T\mathsf{voidgraph}_2 \Rightarrow$ AUTO

10 Discharge $\Rightarrow$ AUTO

Since v_1 is not a singleton, thanks to THM $\mathsf{connectivity}_2$, we can consider a non-cut vertex x_0 of v_1, e_1.

11 $\langle e_2, v_2, v_1\rangle \hookrightarrow T\mathsf{vertexInduced}_0 \Rightarrow$ $e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \subseteq \{\{x,y\} : x \in v_1, y \in v_1 \setminus \{x\}\}$

12 $\big\langle v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}\big\rangle \hookrightarrow T\mathsf{connectivity}_2 \Rightarrow$ $\mathit{Stat10}: \langle \exists u \in v_1 \mid$
$\mathsf{HasSpanningTree}(v_1 \setminus \{u\}, \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid u \notin a\})\rangle$

13 $\langle x_0\rangle \hookrightarrow \mathit{Stat10} \Rightarrow$ $x_0 \in v_1\ \&$
$\mathsf{HasSpanningTree}(v_1 \setminus \{x_0\}, \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\})$

Acyclic extensional orientability of a connected, claw-free graph (contd.)

> Now consider the graph $\mathsf{v}_0, \mathsf{e}_0$ induced by $\mathsf{v}_1, \mathsf{e}_1$ on the strict subset $\mathsf{v}_1 \setminus \{\mathsf{x}_0\}$ of the set of vertices. Before we can utilize the induction hypothesis, which trivially applies to this subgraph, in order to get an acyclic and extensional orientation d_0 of its vertices, we must specify the set of edges of the induced subgraph in two convenient, equivalent ways.

14 Suppose $\Rightarrow$ *Stat11*: $\{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{x}_0 \notin \mathsf{a}\} \neq \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{y} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\}$

15 $\langle \mathsf{a}_1 \rangle \hookrightarrow$ *Stat11* $\Rightarrow$ AUTO

16 Suppose $\Rightarrow$ *Stat12*: $\mathsf{a}_1 \in \{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{x}_0 \notin \mathsf{a}\}$

17 $\langle\ \rangle \hookrightarrow$ *Stat12* $\Rightarrow$ *Stat13*: $\mathsf{a}_1 \in \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}$ & $\mathsf{a}_1 \notin \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{y} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\}$ & $\mathsf{x}_0 \notin \mathsf{a}_1$

18 $\langle \mathsf{x}_4, \mathsf{y}_4, \mathsf{x}_4, \mathsf{y}_4 \rangle \hookrightarrow$ *Stat13* $\Rightarrow$ false

19 Discharge $\Rightarrow$ *Stat14*: $\mathsf{a}_1 \in \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{y} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\}$ & $\mathsf{a}_1 \notin \{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{x}_0 \notin \mathsf{a}\}$ & $\mathsf{a}_1 \in \mathsf{e}_2$

20 $\langle \mathsf{x}_5, \mathsf{y}_5, \{\mathsf{x}_5, \mathsf{y}_5\} \rangle \hookrightarrow$ *Stat14* $\Rightarrow$ *Stat15*: $\{\mathsf{x}_5, \mathsf{y}_5\} \notin \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}$ & $\mathsf{x}_5, \mathsf{y}_5 \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}$

21 $\langle \mathsf{x}_5, \mathsf{y}_5 \rangle \hookrightarrow$ *Stat15* $\Rightarrow$ false

22 Discharge $\Rightarrow$ $\{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{x}_0 \notin \mathsf{a}\} = \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{y} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\}$

23 Suppose $\Rightarrow$ *Stat16*: $\{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{x}_0 \notin \mathsf{a}\} \neq \{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{a} \subseteq \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\}$

24 $\langle \mathsf{a}_3 \rangle \hookrightarrow$ *Stat16* $\Rightarrow$ *Stat17*: $\mathsf{a}_3 \in \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}$ & $\mathsf{x}_0 \notin \mathsf{a}_3$ & $\mathsf{a}_3 \nsubseteq \mathsf{v}_1 \setminus \{\mathsf{x}_0\}$

25 $\langle \mathsf{x}_6, \mathsf{y}_6 \rangle \hookrightarrow$ *Stat17* $\Rightarrow$ false

26 Discharge $\Rightarrow$ $\{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{x}_0 \notin \mathsf{a}\} = \{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{a} \subseteq \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\}$

27 Suppose $\Rightarrow \neg(\mathsf{HasSpanningTree}(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{y} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\})$ & $\mathsf{ClawFreeG}(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{y} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\}))$

28 $\langle \mathsf{v}_1, \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}, \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \rangle \hookrightarrow$ *T*cClawFreeG$_0$ $\Rightarrow$ $\mathsf{ClawFreeG}(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \{\mathsf{a} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\} \mid \mathsf{a} \subseteq \mathsf{v}_1 \setminus \{\mathsf{x}_0\}\})$

29 EQUAL $\Rightarrow$ false

30 Discharge $\Rightarrow$ AUTO

31 $\langle \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat18*: $\langle \exists \mathsf{d} \subseteq (\mathsf{v}_1 \setminus \{\mathsf{x}_0\}) \times (\mathsf{v}_1 \setminus \{\mathsf{x}_0\}) \mid \mathsf{Orientates}(\mathsf{d}, \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{e}_2)$ & $\mathsf{Acyclic}(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d})$ & $\mathsf{Extensional}(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d}) \rangle$

32 $\langle \mathsf{d}_0 \rangle \hookrightarrow$ *Stat18* $\Rightarrow$ $\mathsf{Orientates}(\mathsf{d}_0, \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{e}_2)$ & $\mathsf{Acyclic}(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d}_0)$ & $\mathsf{Extensional}(\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d}_0)$ & $\mathsf{d}_0 \subseteq (\mathsf{v}_1 \setminus \{\mathsf{x}_0\}) \times (\mathsf{v}_1 \setminus \{\mathsf{x}_0\})$

> We first deal with the case when the acyclic, extensional digraph $\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d}_0$ has no sink adjacent to x_0 through e_1 In this case, as suggested by THM cClawFreeG$_1$, we orient the edges incident to x_0 as out-going from x_0, to obtain an extensional acyclic orientation for $\mathsf{v}_1, \mathsf{e}_1$. Note that the neighbors of x_0 through e_1 are $\{\mathsf{t} \in \mathsf{v}_1 \mid \{\mathsf{x}_0, \mathsf{t}\} \in \mathsf{e}_1\}$, hence $\mathsf{d}_1 = \mathsf{d}_0 \cup (\{\mathsf{x}_0\} \times \{\mathsf{t} \in \mathsf{v}_1 \mid \{\mathsf{x}_0, \mathsf{t}\} \in \mathsf{e}_1\})$, although our specification of d_1 will not be so transparent.

33 Suppose $\Rightarrow$ $\{\mathsf{s} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \mid \mathsf{d}_{0|\{\mathsf{s}\}} = \emptyset$ & $\{\mathsf{s}, \mathsf{x}_0\} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}\} = \emptyset$

34 $\langle \mathsf{v}_1, \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d}_0, \mathsf{e}_2 \rangle \hookrightarrow$ *T*orientation$_0$ $\Rightarrow$ $\mathsf{Orientates}(\mathsf{d}_0, \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\})$

35 Loc_def $\Rightarrow$ $\mathsf{d}_1 = \mathsf{d}_0 \cup (\{\mathsf{x}_0\} \times \{\mathsf{t} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \mid \{\mathsf{x}_0, \mathsf{t}\} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}\})$

36 ELEM $\Rightarrow$ $\mathsf{e}_2 \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_2, \mathsf{y} \in \mathsf{v}_2 \setminus \{\mathsf{x}\}\}$ & $\mathsf{v}_1 = \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \cup \{\mathsf{x}_0\}$

37 $\langle \mathsf{v}_1, \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{x}_0, \mathsf{d}_0, \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}, \mathsf{v}_2, \mathsf{d}_1 \rangle \hookrightarrow$ *T*cClawFreeG$_1(\star)$ $\Rightarrow$ *Stat21*: $\mathsf{d}_1 \subseteq \mathsf{v}_1 \times \mathsf{v}_1$ & $\mathsf{Orientates}(\mathsf{d}_1, \mathsf{v}_1, \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\})$ & $\mathsf{Acyclic}(\mathsf{v}_1, \mathsf{d}_1)$ & $\mathsf{Extensional}(\mathsf{v}_1, \mathsf{d}_1)$

38 $\langle \mathsf{v}_1, \mathsf{v}_1, \mathsf{d}_1, \mathsf{e}_2 \rangle \hookrightarrow$ *T*orientation$_0$ $\Rightarrow$ $\mathsf{Orientates}(\mathsf{d}_1, \mathsf{v}_1, \mathsf{e}_2) \leftrightarrow \mathsf{Orientates}(\mathsf{d}_1, \mathsf{v}_1, \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\})$

39 $\langle \mathsf{d}_1 \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ false

40 Discharge $\Rightarrow$ *Stat22*: $\{\mathsf{s} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \mid \mathsf{d}_{0|\{\mathsf{s}\}} = \emptyset$ & $\{\mathsf{s}, \mathsf{x}_0\} \in \mathsf{e}_2 \cap \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_1, \mathsf{y} \in \mathsf{v}_1\}\} \neq \emptyset$

Acyclic extensional orientability of a connected, claw-free graph (contd.)

Next we deal with the case when the acyclic, extensional digraph $\mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{d}_0$ has its sink s_1 adjacent to x_0 through e_1. In this case, as suggested by THM xtensionalization$_2$, we orient the edges incident to x_0 as in-coming to x_0.

41 $\langle \mathsf{s}_1 \rangle \hookrightarrow \mathit{Stat22} \Rightarrow$ $\mathit{Stat23}: \{\mathsf{s}_1, \mathsf{x}_0\} \in \mathsf{e}_2 \;\&\; \mathsf{s}_1 \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \;\&\; \mathsf{d}_{0|\{\mathsf{s}_1\}} = \emptyset$

42 Suppose $\Rightarrow$ $\mathit{Stat24}: \mathsf{s}_1 \notin \{\mathsf{t} \in \mathsf{v}_1 \mid \{\mathsf{x}_0, \mathsf{t}\} \in \mathsf{e}_2\}$

43 $\langle \mathsf{s}_1 \rangle \hookrightarrow \mathit{Stat24} \Rightarrow$ false;

44 Discharge $\Rightarrow$ AUTO

45 $\langle \mathsf{v}_1, \mathsf{v}_1 \setminus \{\mathsf{x}_0\}, \mathsf{x}_0, \mathsf{s}_1, \mathsf{d}_0, \mathsf{e}_2, \mathsf{v}_2 \rangle \hookrightarrow T\mathsf{xtensionalization}_2 \Rightarrow$

$\mathit{Stat25}: \{\mathsf{y} \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \mid [\mathsf{s}_1, \mathsf{y}] \in \mathsf{d}_0\} \neq \emptyset$

46 $\langle \mathsf{y}_7 \rangle \hookrightarrow \mathit{Stat25} \Rightarrow$ $\mathsf{y}_7 \in \mathsf{v}_1 \setminus \{\mathsf{x}_0\} \;\&\; [\mathsf{s}_1, \mathsf{y}_7] \in \mathsf{d}_0$

47 $\langle [\mathsf{s}_1, \mathsf{y}_7], \mathsf{s}_1, \mathsf{y}_7, \mathsf{d}_0, \mathsf{s}_1 \rangle \hookrightarrow T\mathsf{restr}_2 \Rightarrow$ false; Discharge $\Rightarrow$ QED