Overshoot as a Function of Phase Margin

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When an amplifier with a gain ***A(s)*** is put in a feedback loop as shown in Figure 1, the closed loop gain,

$$V_o/V_{in} = A_{CL}$$

$$A_{CL}(s) = \frac{A(s)}{1+\beta A(s)} \quad (1)$$

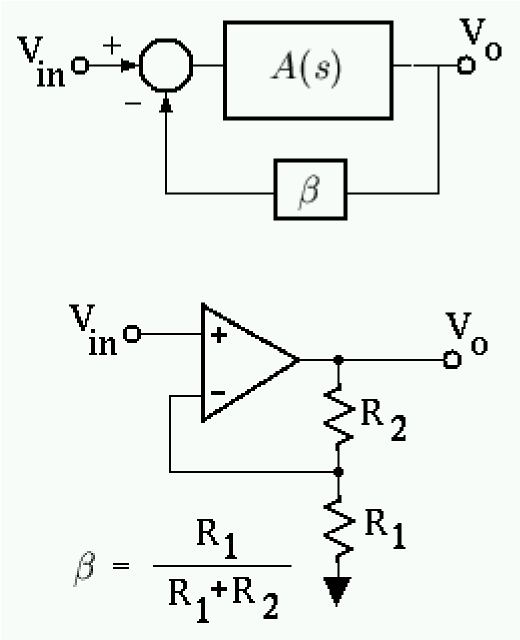

Figure 1

The system is unstable when the loop gain, ***β A (s)***, equals -1. That is, ***β A(s)*** has a magnitude of one and a phase of -180 degrees. An unstable system oscillates. A system close to being unstable has a large ringing overshoot in response to a step input.

The phase margin is a measure of how close the phase of the loop gain is to -180 degrees, when the magnitude of the loop gain is one. The phase margin is the additional phase required to bring the phase of the loop gain to -180 degrees. ***Phase Margin = Phase of loop gain - (-180)***.

The loop gain has a dominant pole at ω_{p1}. Higher order poles can be represented by an equivalent pole at ω_{eq}. The amplifier is approximated by a function with two poles as shown in Equation 2.

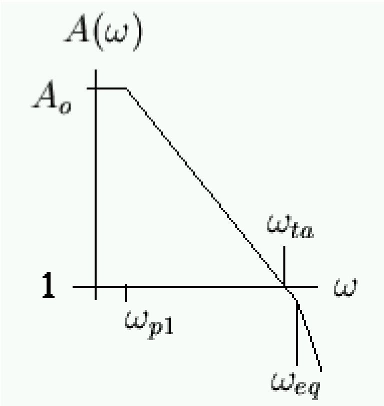

Figure 2 Amplifier frequency response.

$$A(s) = \frac{A_o}{(1+\frac{s}{\omega_{p1}})(1+\frac{s}{\omega_{eq}})} \quad (2)$$

Since for frequencies of interest where the loop gain magnitude is close to unity,

$$\omega > \omega_{p1} \quad (3)$$

And,

$$A(s) = \frac{A_o\omega_{p1}}{s(1+\frac{s}{\omega_{eq}})} \quad (4)$$

Also, it can be shown,

PM	w_t/w_{eq}	Q	%OS
55°	0.700	0.925	13.3%
60°	0.580	0.817	8.7%
65°	0.470	0.717	4.7%
70°	0.360	0.622	1.4%
75°	0.270	0.527	0.008%

Table I

- ***PM*** is the phase margin.
- w_t is the unity gain frequency (rad/sec).
- w_{eq} is the frequency of the equivalent higher order pole

$$\frac{1}{\omega_{eq}} = \frac{1}{\omega_{p2}} + \frac{1}{\omega_{p3}} + \dots \qquad (5)$$

(rad/sec).

- ***Q*** is the system Quality factor.
- ***OS*** is the Over Shoot.

Defining ω_{ta},

$$\omega_{p1} A_o = \omega_{ta} \qquad (6)$$

For frequencies of interest (frequencies close to the unity gain frequency), the amplifier gain can be written,

$$A(s) = \frac{\omega_{ta}}{s(1 + \frac{s}{\omega_{eq}})} \qquad (7)$$

Plugging Equation 7 into Equation 1 results in the following expression for the closed loop gain.

$$A_{CL}(s) = \frac{1}{\beta} \frac{1}{1 + \frac{s}{\omega_{ta\beta}} + \frac{s^2}{\omega_{ta}\beta\omega_{eq}}} \qquad (8)$$

Equation 8 is the transfer function for a second order system. The general form for the response of a second order system, where system properties are described by its ***Q*** and resonant frequence w_o, is shown in Equation 10.

$$A_{CL}(s) = \frac{K}{1 + \frac{s}{Q\omega_o} + \frac{s^2}{\omega_o^2}} \qquad (9)$$

By comparing Equation 8 to Equation 9 we can get an expression for the resonant frequency and **Q** of the amplifier closed loop gain. (Equate coefficients of like powers of s in the dominators.)

$$\omega_o = \sqrt{\beta \omega_{ta} \omega_{eq}} \qquad (10)$$

$$Q = \sqrt{\frac{\beta \omega_{ta}}{\omega_{eq}}} \qquad (11)$$

The loop gain is the feedback factor ß multiplied by the amplifier gain ***A(s)***.

$$\beta A(s) = \frac{\beta \omega_{ta}}{s(1 + \frac{s}{\omega_{eq}})} \qquad (12)$$

The phase margin is a function of the phase of the loop gain at the frequency where the magnitude of the loop gain is unity.

$$\beta A(\omega_t) = 1 \qquad (13)$$

where ω_t is the loop gain unity gain frequency. It follows from Equations 12 and 13 that,

$$\beta^2\omega_{ta}^2 = \omega_t^2\left[1+\frac{\omega_t^2}{\omega_{eq}^2}\right] \qquad (14)$$

Also, solving for w_{ta} and dividing by w_{eq},

$$\frac{\omega_{ta}}{\omega_{eq}} = \frac{1}{\beta}\frac{\omega_t}{\omega_{eq}}\sqrt{1+\frac{\omega_t^2}{\omega_{eq}^2}} \qquad (15)$$

It follows from Equations 11 and 15 that,

$$Q = \sqrt{\frac{\omega_t}{\omega_{eq}}\left(1+\frac{\omega_t^2}{\omega_{eq}^2}\right)^{\frac{1}{2}}} \qquad (16)$$

The phase of the loop gain (Equation 13) is.

$$\textbf{\textit{Phase of loop gain}} = -90^o - tan^{-1}\left(\frac{\omega_t}{\omega_{eq}}\right) \qquad (17)$$

The phase margin is the additional phase required to bring the phase of the loop gain to -180 degrees. ***Phase Margin = Phase of loop gain - (-180)***.

$$\textbf{\textit{Phase Margin}} = 90^o - tan^{-1}\left(\frac{\omega_t}{\omega_{eq}}\right) \qquad (18)$$

A well known property of second order systems is that the percent overshoot is a function of the Q and is given by,

$$\%OS = 100e^{-\frac{\pi}{\sqrt{4Q^2-1}}} \qquad (19)$$

Both phase margin (Equation 18) and Q (Equation 16) are a function of w_t / w_{eq}. This allows us to use Equation 19 to create tables and plots of percent overshoot as a function of phase margin. As shown in Figures 3 and 4, and in Table I.

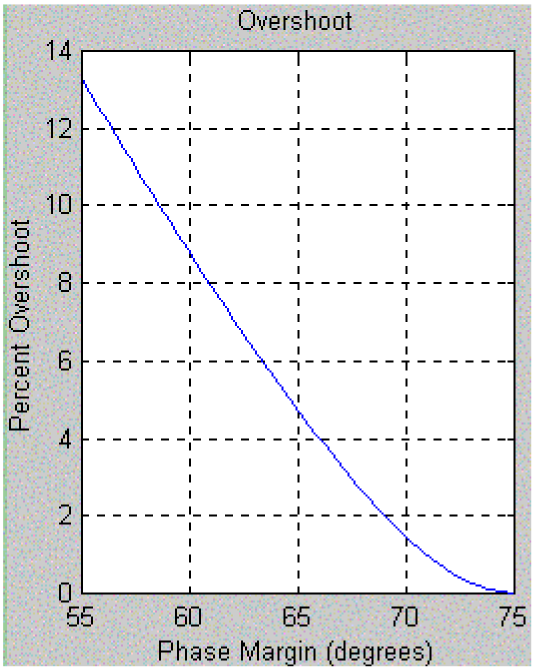

Figure 3 Overshoot as a function of phase margin.
Plot generated using [MATLAB code.]

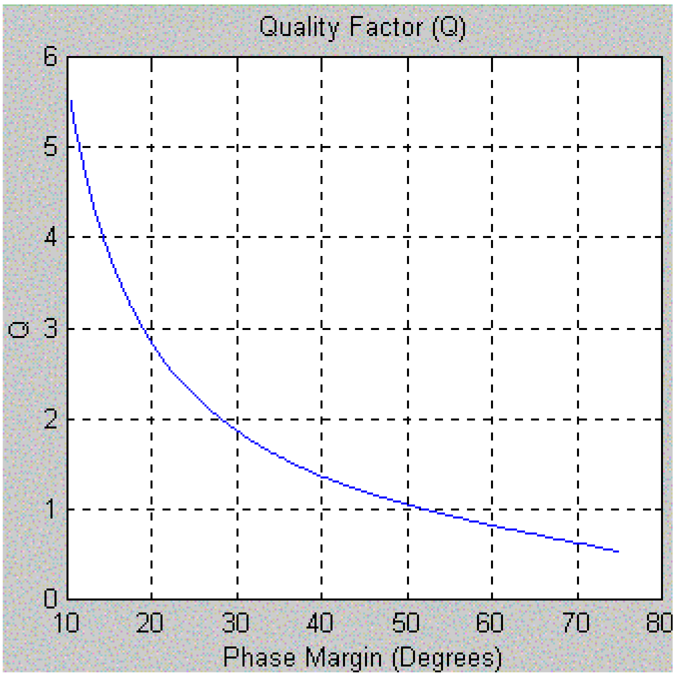

Figure 4 **Q** as a function of phase margin.
Plot generated using [MATLAB code.]